A KEY

TO

ALGEBRA

PART I.

BY

J. HAMBLIN SMITH, M.A.

GONVILLE AND CAIUS COLLEGE,

AND LATE LECTURER AT ST. PETER'S COLLEGE, CAMBRIDGE

THIRD EDITION

RIVINGTONS

WATERLOO PLACE, LONDON

Oxford and Cambridge

MDCCCLXXX

PREFACE.

THIS Book is published at the request of many Teachers. It is intended to assist, *first*, Masters who cannot devote much time to the examination of the work of their pupils, and *secondly*, Students who are unable to obtain the help of competent instructors. In working the Exercises, the simplest and most obvious methods of solution have in all cases been adopted.

J. HAMBLIN SMITH.

CAMBRIDGE, *May* 1874.

[C—414]

TABLE OF REFERENCE TO EXERCISES IN ELEMENTARY ALGEBRA.

10. $b-[b-a-b-\{b-(b-a+b)\}]=b-[b-a-b-\{b-b+a-b\}]$
$=b-[b-a-b-b+b-a+b]=b-b+a+b+b-b+a-b=2a.$

11. $2c-6a+b-\{c-5a-2b-a+3b\}=2c-6a+b-c+5a+2b+a-3b=c.$

12. $2x-\{a-(2a-[3a-(4a-[5a-6a+x])])\}$
$=2x-\{a-(2a-[3a-(4a-5a+6a-x)])\}$
$=2x-\{a-(2a-[3a-4a+5a-6a+x])\}$
$=2x-\{a-(2a-3a+4a-5a+6a-x)\}$
$=2x-\{a-2a+3a-4a+5a-6a+x\}$
$=2x-a+2a-3a+4a-5a+6a-x=x+3a.$

13. $25a-19b-[3b-\{4a-5b+6c\}]=25a-19b-[3b-4a+5b-6c]$
$=25a-19b-3b+4a-5b+6c=29a-27b+6c.$

VIII.

1.
$$\begin{array}{l} x+3 \\ x+9 \\ \hline x^2+3x \\ \quad +9x+27 \\ \hline x^2+12x+27 \end{array}$$

2.
$$\begin{array}{l} x+15 \\ x-7 \\ \hline x^2+15x \\ \quad -7x-105 \\ \hline x^2+8x-105 \end{array}$$

3.
$$\begin{array}{l} x-12 \\ x+10 \\ \hline x^2-12x \\ \quad +10x-120 \\ \hline x^2-2x-120 \end{array}$$

4.
$$\begin{array}{l} x-8 \\ x-7 \\ \hline x^2-8x \\ \quad -7x+56 \\ \hline x^2-15x+56 \end{array}$$

5.
$$\begin{array}{l} a-3 \\ a-5 \\ \hline a^2-3a \\ \quad -5a+15 \\ \hline a^2-8a+15 \end{array}$$

6.
$$\begin{array}{l} y-6 \\ y+13 \\ \hline y^2-6y \\ \quad +13y-78 \\ \hline y^2+7y-78 \end{array}$$

7.
$$\begin{array}{l} x^2-4 \\ x^2+5 \\ \hline x^4-4x^2 \\ \quad +5x^2-20 \\ \hline x^4+x^2-20 \end{array}$$

8.
$$\begin{array}{l} x^2-6x+9 \\ x^2-6x+5 \\ \hline x^4-6x^3+9x^2 \\ \quad -6x^3+36x^2-54x \\ \qquad\qquad +5x^2-30x+45 \\ \hline x^4-12x^3+50x^2-84x+45 \end{array}$$

9.
$$\begin{array}{l} x^2+5x-3 \\ x^2-5x-3 \\ \hline x^4+5x^3-3x^2 \\ \quad -5x^3-25x^2+15x \\ \qquad\quad -3x^2-15x+9 \\ \hline x^4 \qquad -31x^2 \qquad +9 \end{array}$$

10.
$$\begin{array}{l} a^3-3a+2 \\ a^3-3a^2+2 \\ \hline a^6-3a^4+2a^3 \\ \quad -3a^5+9a^3-6a^2 \\ \qquad\quad +2a^3-6a+4 \\ \hline a^6-3a^5-3a^4+13a^3-6a^2-6a+4 \end{array}$$

11.
$$\begin{array}{l} x^2-x+1 \\ x^2+x-1 \\ \hline x^4-x^3+x^2 \\ \quad +x^3-x^2+x \\ \qquad\quad -x^2+x-1 \\ \hline x^4 \qquad -x^2+2x-1 \end{array}$$

12.
$$\begin{array}{l} x^2+xy+y^2 \\ x^2-xy+y^2 \\ \hline x^4+x^3y+x^2y^2 \\ \quad -x^3y-x^2y^2-xy^3 \\ \qquad\quad +x^2y^2+xy^3+y^4 \\ \hline x^4 \qquad +x^2y^2 \qquad +y^4 \end{array}$$

13.
$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \qquad\qquad -y^3 \end{array}$$

14.
$$\begin{array}{l} a^2-x^2 \\ a^4+a^2x^2+x^4 \\ \hline a^6-a^4x^2 \\ \quad +a^4x^2-a^2x^4 \\ \qquad\quad +a^2x^4-x^6 \\ \hline a^6 \qquad\qquad -x^6 \end{array}$$

15.
$$\begin{array}{l} x^3-3x^2+3x-1 \\ x^2+3x+1 \\ \hline x^5-3x^4+3x^3-x^2 \\ \quad +3x^4-9x^3+9x^2-3x \\ \qquad\quad +x^3-3x^2+3x-1 \\ \hline x^5 \qquad -5x^3+5x^2 \qquad -1 \end{array}$$

16.
$$\begin{array}{l} x^3+3x^2y+9xy^2+27y^3 \\ x-3y \\ \hline x^4+3x^3y+9x^2y^2+27xy^3 \\ \quad -3x^3y-9x^2y^2-27xy^3-81y^4 \\ \hline x^4 \qquad\qquad\qquad -81y^4 \end{array}$$

17.
$$\begin{array}{l} a^3+2a^2b+4ab^2+8b^3 \\ a-2b \\ \hline a^4+2a^3b+4a^2b^2+8ab^3 \\ \quad -2a^3b-4a^2b^2-8ab^3-16b^4 \\ \hline a^4 \qquad\qquad\qquad -16b^4 \end{array}$$

18.
$$\begin{array}{l} 8a^3+4a^2b+2ab^2+b^3 \\ 2a-b \\ \hline 16a^4+8a^3b+4a^2b^2+2ab^3 \\ \quad -8a^3b-4a^2b^2-2ab^3-b^4 \\ \hline 16a^4 \qquad\qquad\qquad -b^4 \end{array}$$

19.
$$\begin{array}{l}
a^3 - 2a^2b + 3ab^2 + 4b^3 \\
a^2 - 2ab - 3b^2 \\
\hline
a^5 - 2a^4b + 3a^3b^2 + 4a^2b^3 \\
\quad - 2a^4b + 4a^3b^2 - 6a^2b^3 - 8ab^4 \\
\qquad\qquad - 3a^3b^2 + 6a^2b^3 - 9ab^4 - 12b^5 \\
\hline
a^5 - 4a^4b + 4a^3b^2 + 4a^2b^3 - 17ab^4 - 12b^5
\end{array}$$

20.
$$\begin{array}{l}
a^3 + 3a^2b - 2ab^2 + 3b^3 \\
a^2 + 2ab - 3b^2 \\
\hline
a^5 + 3a^4b - 2a^3b^2 + 3a^2b^3 \\
\quad + 2a^4b + 6a^3b^2 - 4a^2b^3 + 6ab^4 \\
\qquad\qquad - 3a^3b^2 - 9a^2b^3 + 6ab^4 - 9b^5 \\
\hline
a^5 + 5a^4b + a^3b^2 - 10a^2b^3 + 12ab^4 - 9b^5
\end{array}$$

21.
$$\begin{array}{l}
a^2 - 2ax + 4x^2 \\
a^2 + 2ax + 4x^2 \\
\hline
a^4 - 2a^3x + 4a^2x^2 \\
\quad + 2a^3x - 4a^2x^2 + 8ax^3 \\
\qquad\qquad + 4a^2x^2 - 8ax^3 + 16x^4 \\
\hline
a^4 \qquad + 4a^2x^2 \qquad + 16x^4
\end{array}$$

22.
$$\begin{array}{l}
9a^2 + 3ax + x^2 \\
9a^2 - 3ax + x^2 \\
\hline
81a^4 + 27a^3x + 9a^2x^2 \\
\qquad - 27a^3x - 9a^2x^2 - 3ax^3 \\
\qquad\qquad\qquad + 9a^2x^2 + 3ax^3 + x^4 \\
\hline
81a^4 \qquad + 9a^2x^2 \qquad + x^4
\end{array}$$

23.
$$\begin{array}{l}
x^4 - 2ax^2 + 4a^2 \\
x^4 + 2ax^2 + 4a^2 \\
\hline
x^8 - 2ax^6 + 4a^2x^4 \\
\quad + 2ax^6 - 4a^2x^4 + 8a^3x^2 \\
\qquad\qquad + 4a^2x^4 - 8a^3x^2 + 16a^4 \\
\hline
x^8 \qquad + 4a^2x^4 \qquad + 16a^4
\end{array}$$

24.
$$\begin{array}{l}
a^2 + b^2 + c^2 - ab - ac - bc \\
a \ + b \ + c \\
\hline
a^3 + ab^2 + ac^2 - a^2b - a^2c - abc \\
\quad + a^2b + b^3 + bc^2 - ab^2 - abc - b^2c \\
\qquad\qquad + a^2c + b^2c + c^3 - abc - ac^2 - bc^2 \\
\hline
a^3 + b^3 + c^3 - 3abc
\end{array}$$

25.
$$\begin{array}{llllll}
x^2 + 4xy & + 5y^2 & & & & \\
x^3 - 3x^2y & - 2xy^2 & + 3y^3 & & & \\
\hline
x^5 + 4x^4y & + 5x^3y^2 & & & & \\
\quad - 3x^4y & - 12x^3y^2 & - 15x^2y^3 & & & \\
 & - 2x^3y^2 & - 8x^2y^3 & - 10xy^4 & & \\
 & & + 3x^2y^3 & + 12xy^4 & + 15y^5 & \\
\hline
x^5 + x^4y & - 9x^3y^2 & - 20x^2y^3 & + 2xy^4 & + 15y^5 &
\end{array}$$

26.
$$\begin{array}{lllllll}
ab + cd & + ac & + bd & & & & \\
ab + cd & - ac & - bd & & & & \\
\hline
a^2b^2 & + abcd & + a^2bc & + ab^2d & & & \\
 & + abcd & + c^2d^2 & + ac^2d & + bcd^2 & & \\
 & & - a^2bc & - ac^2d & - a^2c^2 & - abcd & \\
 & & & - ab^2d & - bcd^2 & - abcd & - b^2d^2 \\
\hline
a^2b^2 & & + c^2d^2 & & - a^2c^2 & & - b^2d^2
\end{array}$$

27. $(x-a)(x+a)(x^2+a^2)(x^4+a^4) = (x^2-a^2)(x^2+a^2)(x^4+a^4)$
$= (x^4-a^4)(x^4+a^4) = x^8 - a^8.$

28. $(x-a)(x+b)(x-c) = (x^2 - ax + bx - ab)(x-c)$
$= x^3 - ax^2 + bx^2 - cx^2 - abx + acx - bcx + abc.$

29. $(1-x)(1+x)(1+x^2)(1+x^4) = (1-x^2)(1+x^2)(1+x^4)$
$= (1-x^4)(1+x^4) = 1 - x^8.$

30. First multiply $x^2 + xy + y^2$ by $x - y$; the result is $x^3 - y^3$.
Then multiply $x^2 - xy + y^2$ by $x + y$; the result is $x^3 + y^3$.
Then multiply $x^3 - y^3$ by $x^3 + y^3$; the result is $x^6 - y^6$.

31. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$= (a^2-x^2)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$= (a^4-x^4)(a^4+x^4)(a^8+x^8)$
$= (a^8-x^8)(a^8+x^8) = a^{16} - x^{16}.$

32. $(x-5)(x-6)(x+7) = (x^2 - 11x + 30)(x+7) = x^3 - 4x^2 - 47x + 210$;
therefore the coefficient of x is -47.

33. $(x+8)(x+3)(x-2)=(x^2+11x+24)(x-2)=x^3+9x^2+2x-48$;
therefore the coefficient of x is 2.

34. $(x-2)(x-3)(x+4)=(x^2-5x+6)(x+4)=x^3-x^2-14x+24$;
therefore the coefficient of x is -14.

35. $(x-a)(x-b)(x-c)=(x^2-ax-bx+ab)(x-c)$
$=x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc$;
therefore the coefficient of x is $ab+ac+bc$.

36. $(x^2+3x-2)(x^2-3x+2)(x^4-5)=(x^4-9x^2+12x-4)(x^4-5)$
$=x^8-9x^6+12x^5-9x^4+45x^2-60x+20$;
therefore the coefficient of x is -60.

37. $(x^2-x+1)(x^2+x-1)(x^4-x^2+1)=(x^4-x^2+2x-1)(x^4-x^2+1)$
$=x^8-2x^6+2x^5+x^4-2x^3+2x-1$;
therefore the coefficient of x is 2.

38. $(x^2-mx+1)(x^2-mx-1)(x^4-m^2x-1)$
$=(x^4-2mx^3+m^2x^2-1)(x^4-m^2x-1)=x^8-2mx^7+m^2x^6-m^2x^5$
$+2m^3x^4-2x^4-m^4x^3+2mx^3-m^2x^2+m^2x+1$;
therefore the coefficient of x is m^2.

IX.

8.
$$\begin{array}{l}
-a^3-a^2-a \\
-a-1 \\
\hline
a^4+a^3+a^2 \\
\quad +a^3+a^2+a \\
\hline
a^4+2a^3+2a^2+a
\end{array}$$

9.
$$\begin{array}{l}
3x^2y-5xy^2+4y^3 \\
-2x-3y \\
\hline
-6x^3y+10x^2y^2-8xy^3 \\
\qquad\qquad -9x^2y^2+15xy^3-12y^4 \\
\hline
-6x^3y+x^2y^2+7xy^3-12y^4
\end{array}$$

10.
$$\begin{array}{l}
-5m^2-6mn+7n^2 \\
-m+n \\
\hline
5m^3+6m^2n-7mn^2 \\
\qquad -5m^2n-6mn^2+7n^3 \\
\hline
5m^3+m^2n-13mn^2+7n^3
\end{array}$$

11.
$$\begin{array}{l}
13r^2-17r-45 \\
-r-3 \\
\hline
-13r^3+17r^2+45r \\
\qquad -39r^2+51r+135 \\
\hline
-13r^3-22r^2+96r+135
\end{array}$$

12. $$\begin{array}{l} 7x^3-8x^2z-9z^2 \\ -x-z \\ \hline -7x^4+8x^3z+9xz^2 \\ \qquad -7x^3z+8x^2z^2+9z^3 \\ \hline -7x^4+x^3z+8x^2z^2+9xz^2+9z^3 \end{array}$$

13. $$\begin{array}{l} -x^5+x^4y-x^3y^2 \\ -x-y \\ \hline x^6-x^5y+x^4y^2 \\ \qquad +x^5y-x^4y^2+x^3y^3 \\ \hline x^6 \qquad\qquad +x^3y^3 \end{array}$$

14. $$\begin{array}{l} -x^3-x^2y-xy^2-y^3 \\ -x-y \\ \hline x^4+x^3y+x^2y^2+xy^3 \\ \qquad +x^3y+x^2y^2+xy^3+y^4 \\ \hline x^4+2x^3y+2x^2y^2+2xy^3+y^4 \end{array}$$

X.

12. $(x^2+2x-3)^2=x^4+4x^2+9+4x^3-6x^2-12x=x^4+4x^3-2x^2-12x+9.$

13. $(x^2-6x+7)^2=x^4+36x^2+49-12x^3+14x^2-84x$
$=x^4-12x^3+50x^2-84x+49.$

14. $(2x^2-7x+9)^2=4x^4+49x^2+81-28x^3+36x^2-126x$
$=4x^4-28x^3+85x^2-126x+81.$

16. $(x^4-4x^2y^2+y^4)^2=x^8+16x^4y^4+y^8-8x^6y^2+2x^4y^4-8x^2y^6$
$=x^8-8x^6y^2+18x^4y^4-8x^2y^6+y^8.$

27. $(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$
$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$

28. $(a-b-c)^3=(a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3$
$=a^3-3a^2b+3ab^2-b^3-3a^2c+6abc-3b^2c+3ac^2-3bc^2-c^3.$

29. $(m+n)(m-n)(m+n)(m-n)=(m^2-n^2)(m^2-n^2)$
$=m^4-2m^2n^2+n^4.$

30. $(m^2+2mn+n^2)(m^2-n^2)=m^4+2m^3n+m^2n^2-m^2n^2-2mn^3-n^4$
$=m^4+2m^3n-2mn^3-n^4.$

XIV.

1.
$$\begin{array}{l} x+10)x^2+15x+50(x+5 \\ \underline{x^2+10x} \\ \qquad 5x+50 \\ \qquad \underline{5x+50} \end{array}$$

2.
$$\begin{array}{l} x-7)x^2-17x+70(x-10 \\ \underline{x^2-7x} \\ \qquad -10x+70 \\ \qquad \underline{-10x+70} \end{array}$$

3.
$$\begin{array}{l} x-3)x^2+x-12(x+4 \\ \underline{x^2-3x} \\ \qquad 4x-12 \\ \qquad \underline{4x-12} \end{array}$$

4.
$$\begin{array}{l} x+1)x^2+13x+12(x+12 \\ \underline{x^2+x} \\ \qquad 12x+12 \\ \qquad \underline{12x+12} \end{array}$$

5.
$$\begin{array}{l} x+6)x^3+13x^2+54x+72(x^2+7x+12 \\ \underline{x^3+6x^2} \\ \qquad 7x^2+54x \\ \qquad \underline{7x^2+42x} \\ \qquad\qquad 12x+72 \\ \qquad\qquad \underline{12x+72} \end{array}$$

6.
$$\begin{array}{l} x+1)x^3+x^2-x-1(x^2-1 \\ \underline{x^3+x^2} \\ \qquad -x-1 \\ \qquad \underline{-x-1} \end{array}$$

7.
$$\begin{array}{l} x+1)x^3+2x^2+2x+1(x^2+x+1 \\ \underline{x^3+x^2} \\ \qquad x^2+2x \\ \qquad \underline{x^2+x} \\ \qquad\qquad x+1 \\ \qquad\qquad \underline{x+1} \end{array}$$

8.
$$\begin{array}{l} x^2+3x+1)x^5-5x^3+7x^2+6x+1(x^3-3x^2+3x+1 \\ \underline{x^5+3x^4+x^3} \\ \qquad -3x^4-6x^3+7x^2 \\ \qquad \underline{-3x^4-9x^3-3x^2} \\ \qquad\qquad 3x^3+10x^2+6x \\ \qquad\qquad \underline{3x^3+9x^2+3x} \\ \qquad\qquad\qquad x^2+3x+1 \\ \qquad\qquad\qquad \underline{x^2+3x+1} \end{array}$$

9. $x^2-2x-1)x^4-4x^3+2x^2+4x+1(x^2-2x-1$

$$\begin{array}{l}
\underline{x^4-2x^3-x^2} \\
\quad -2x^3+3x^2+4x \\
\quad \underline{-2x^3+4x^2+2x} \\
\qquad\qquad -x^2+2x+1 \\
\qquad\qquad \underline{-x^2+2x+1}
\end{array}$$

10. $x^2-2x+1)x^4-4x^3+6x^2-4x+1(x^2-2x+1$

$$\begin{array}{l}
\underline{x^4-2x^3+x^2} \\
\quad -2x^3+5x^2-4x \\
\quad \underline{-2x^3+4x^2-2x} \\
\qquad\qquad\qquad x^2-2x-1 \\
\qquad\qquad\qquad \underline{x^2-2x+1}
\end{array}$$

11. $x^2+x-1)x^4-x^2+2x-1(x^2-x+1$

$$\begin{array}{l}
\underline{x^4+x^3-x^2} \\
\quad -x^3+2x-1 \\
\quad \underline{-x^3-x^2+x} \\
\qquad\qquad x^2+x-1 \\
\qquad\qquad \underline{x^2+x-1}
\end{array}$$

12. $x+2)x^4-4x^2+8x+16(x^3-2x^2+8$

$$\begin{array}{l}
\underline{x^4+2x^3} \\
\quad -2x^3-4x^2 \\
\quad \underline{-2x^3-4x^2} \\
\qquad\qquad 8x+16 \\
\qquad\qquad \underline{8x+16}
\end{array}$$

13. $x+4y)x^3+4x^2y+3xy^2+12y^3(x^2+3y^2$

$$\underline{x^3+4x^2y}$$

$$3xy^2+12y^3$$

$$\underline{3xy^2+12y^3}$$

14. $a+b)a^4+4a^3b+6a^2b^2+4ab^3+b^4(a^3+3a^2b+3ab^2+b^3$

$$\underline{a^4+a^3b}$$

$$3a^3b+6a^2b^2$$

$$\underline{3a^3b+3a^2b^2}$$

$$3a^2b^2+4ab^3$$

$$\underline{3a^2b^2+3ab^3}$$

$$ab^3+b^4$$

$$\underline{ab^3+b^4}$$

15. $a\quad b)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5(a^4-4a^3b+6a^2b^2-4ab^3+b^4$

$$\underline{a^5-a^4b}$$

$$-4a^4b+10a^3b^2$$

$$\underline{-4a^4b+4a^3b^2}$$

$$6a^3b^2-10a^2b^3$$

$$\underline{6a^3b^2-6a^2b^3}$$

$$-4a^2b^3+5ab^4$$

$$\underline{-4a^2b^3+4ab^4}$$

$$ab^4-b^5$$

$$\underline{ab^4-b^5}$$

16. $x^2-6x+9)x^4-12x^3+50x^2-84x+45(x^2-6x+5$

$$\underline{x^4-6x^3+9x^2}$$

$$-6x^3+41x^2-84x$$

$$\underline{-6x^3+36x^2-54x}$$

$$5x^2-30x+45$$

$$\underline{5x^2-30x+45}$$

17. $a^2-2ab-3b^2)\,a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5\,(a^3-2a^2b+3ab^2+4b^3$

$$\underline{a^5-2a^4b-3a^3b^2}$$
$$-2a^4b+7a^3b^2+4a^2b^3$$
$$\underline{-2a^4b+4a^3b^2+6a^2b^3}$$
$$3a^3b^2-2a^2b^3-17ab^4$$
$$\underline{3a^3b^2-6a^2b^3-9ab^4}$$
$$4a^2b^3-8ab^4-12b^5$$
$$\underline{4a^2b^3-8ab^4-12b^5}$$

18. $2ax^2-3a^2x+a^3)\,4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6\,(2ax^2-3a^2x+a^3$

$$\underline{4a^2x^4-6a^3x^3+2a^4x^2}$$
$$-6a^3x^3+11a^4x^2-6a^5x$$
$$\underline{-6a^3x^3+9a^4x^2-3a^5x}$$
$$2a^4x^2-3a^5x+a^6$$
$$\underline{2a^4x^2-3a^5x+a^6}$$

19. $x^2+x-1)\,x^4-x^2+2x-1\,(x^2-x+1$

$$\underline{x^4+x^3-x^2}$$
$$-x^3+2x-1$$
$$\underline{-x^3-x^2+x}$$
$$x^2+x-1$$
$$\underline{x^2+x-1}$$

20. $x^2+2a^2)\,x^4+a^2x^2-2a^4\,(x^2-a^2$

$$\underline{x^4+2a^2x^2}$$
$$-a^2x^2-2a^4$$
$$\underline{-a^2x^2-2a^4}$$

21. $x-15y)\,x^2-13xy-30y^2\,(x+2y$

$$\underline{x^2-15xy}$$
$$2xy-30y^2$$
$$\underline{2xy-30y^2}$$

22. $x+y)\,x^5+y^5\,(x^4-x^3y+x^2y^2-xy^3+y^4$

$$\underline{x^5+x^4y}$$
$$-x^4y+y^5$$
$$\underline{-x^4y-x^3y^2}$$
$$x^3y^2+y^5$$
$$\underline{x^3y^2+x^2y^3}$$
$$-x^2y^3+y^5$$
$$\underline{-x^2y^3-xy^4}$$
$$xy^4+y^5$$
$$\underline{xy^4+y^5}$$

23. $$\begin{array}{l} x-y)\,x^6-y^6\,(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5 \\ x^6-x^5y \\ x^5y-y^6 \\ x^5y-x^4y^2 \\ x^4y^2-y^6 \\ x^4y^2-x^3y^3 \\ x^3y^3-y^6 \\ x^3y^3-x^2y^4 \\ x^2y^4-y^6 \\ x^2y^4-xy^5 \\ xy^5-y^6 \\ xy^5-y^6 \end{array}$$

24. $$\begin{array}{l} a-b+c)\,a^2-b^2+2bc-c^2\,(a+b-c \\ a^2-ab+ac \\ ab-b^2-ac+2bc-c^2 \\ ab-b^2+bc \\ -ac+bc-c^2 \\ -ac+bc-c^2 \end{array}$$

25. $$\begin{array}{l} -1+b)\,b-3b^2+3b^3-b^4\,(-b+2b^2-b^3 \\ b-b^2 \\ -2b^2+3b^3 \\ -2b^2+2b^3 \\ b^3-b^4 \\ b^3-b^4 \end{array}$$

26. $$\begin{array}{l} a+b-c-d)\,a^2-2ad+2bc-b^2-c^2+d^2\,(a-b+c-d \\ a^2+ab-ac-ad \\ -ab-b^2+ac-ad+2bc-c^2+d^2 \\ -ab-b^2+bc+bd \\ ac+bc-bd-ad-c^2+d^2 \\ ac+bc-c^2-cd \\ -ad-bd+cd+d^2 \\ ad-bd+cd+d^2 \end{array}$$

27. $$x+y+z)\,x^3+y^3+z^3-3xyz\,(x^2-xy-xz+y^2-yz+z^2$$
$$\underline{x^3+x^2y+x^2z}$$
$$-x^2y-x^2z-3xyz+y^3+z^3$$
$$\underline{-x^2y-xy^2-xyz}$$
$$-x^2z+xy^2-2xyz+y^3+z^3$$
$$\underline{-x^2z-xyz-xz^2}$$
$$xy^2-xyz+xz^2+y^3+z^3$$
$$\underline{xy^2+y^3+y^2z}$$
$$-xyz-y^2z+xz^2+z^3$$
$$\underline{-xyz-y^2z-yz^2}$$
$$xz^2+yz^2+z^3$$
$$\underline{xz^2+yz^2+z^3}$$

28. $$x^3+y^2)\,x^{15}+y^{10}\,(x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8$$
$$\underline{x^{15}+x^{12}y^2}$$
$$-x^{12}y^2+y^{10}$$
$$\underline{-x^{12}y^2-x^9y^4}$$
$$x^9y^4+y^{10}$$
$$\underline{x^9y^4+x^6y^6}$$
$$-x^6y^6+y^{10}$$
$$\underline{-x^6y^6-x^3y^8}$$
$$x^3y^8+y^{10}$$
$$\underline{x^3y^8+y^{10}}$$

29. $$p-q+3r)\,p^2+pq+2pr-2q^2+7qr-3r^2\,(p+2q-r$$
$$\underline{p^2-pq+3pr}$$
$$2pq-pr-2q^2+7qr$$
$$\underline{2pq-2q^2+6qr}$$
$$-pr+qr-3r^2$$
$$\underline{-pr+qr-3r^2}$$

30. $a^4 + a^3b + a^2b^2 + ab^3 + b^4)\,a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8\,(a^4 - a^3b + a^2b^2 - ab^3 + b^4$

$$\underline{a^8 + a^7b + a^6b^2 + a^5b^3 + a^4b^4}$$
$$-a^7b - a^5b^3 + a^2b^6 + b^8$$
$$\underline{-a^7b - a^6b^2 - a^5b^3 - a^4b^4 - a^3b^5}$$
$$a^6b^2 + a^4b^4 + a^3b^5 + a^2b^6 + b^8$$
$$\underline{a^6b^2 + a^5b^3 + a^4b^4 + a^3b^5 + a^2b^6}$$
$$-a^5b^3 + b^8$$
$$\underline{-a^5b^3 - a^4b^4 - a^3b^5 - a^2b^6 - ab^7}$$
$$a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8$$
$$\underline{a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8}$$

31. $x^4 - x^3y + x^2y^2 - xy^3 + y^4)\,x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8\,(x^4 + x^3y + x^2y^2 + xy^3 + y^4$

$$\underline{x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4}$$
$$x^7y + x^5y^3 + x^2y^6 + y^8$$
$$\underline{x^7y - x^6y^2 + x^5y^3 - x^4y^4 + x^3y^5}$$
$$x^6y^2 + x^4y^4 - x^3y^5 + x^2y^6 + y^8$$
$$\underline{x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6}$$
$$x^5y^3 + y^8$$
$$\underline{x^5y^3 - x^4y^4 + x^3y^5 - x^2y^6 + xy^7}$$
$$x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8$$
$$\underline{x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8}$$

32. $2x^2 + 3x + 2)\,4x^5 - x^3 + 4x\,(2x^3 - 3x^2 + 2x$

$$\underline{4x^5 + 6x^4 + 4x^3}$$
$$-6x^4 - 5x^3 + 4x$$
$$\underline{-6x^4 - 9x^3 - 6x^2}$$
$$4x^3 + 6x^2 + 4x$$
$$\underline{4x^3 + 6x^2 + 4x}$$

33. $a-3)\,a^5-243\,(a^4+3a^3+9a^2+27a+81$

$\underline{a^5-3a^4}$

$3a^4-243$

$\underline{3a^4-9a^3}$

$9a^3-243$

$\underline{9a^3-27a^2}$

$27a^2-243$

$\underline{27a^2-81a}$

$81a-243$

$\underline{81a-243}$

34. $k^3-1)\,k^{10}-k\,(k^7+k^4+k$

$\underline{k^{10}-k^7}$

k^7-k

$\underline{k^7-k^4}$

k^4-k

$\underline{k^4-k}$

35. $x+4)\,x^3-5x^2-46x-40\,(x^2-9x-10$

$\underline{x^3+4x^2}$

$-9x^2-46x$

$\underline{-9x^2-36x}$

$-10x-40$

$\underline{-10x-40}$

36. $2x-3a)\,48x^3-76ax^2-64a^2x+105a^3\,(24x^2-2ax-35a^2$

$\underline{48x^3-72ax^2}$

$-4ax^2-64a^2x$

$\underline{-4ax^2+6a^2x}$

$-70a^2x+105a^3$

$\underline{-70a^2x+105a^3}$

37. $3x^2-4x+5)\,18x^4-45x^3+82x^2-67x+40\,(6x^2-7x+8$

$\underline{18x^4-24x^3+30x^2}$

$-21x^3+52x^2-67x$

$\underline{-21x^3+28x^2-35x}$

$24x^2-32x+40$

$\underline{24x^2-32x+40}$

38. $2x-3a)16x^4-72a^2x^2+81a^4(8x^3+12ax^2-18a^2x-27a^3$

$$\begin{array}{l}16x^4-24ax^3\\ \hline 24ax^3-72a^2x^2\\ 24ax^3-36a^2x^2\\ \hline -36a^2x^2+81a^4\\ -36a^2x^2+54a^3x\\ \hline -54a^3x+81a^4\\ -54a^3x+81a^4\\ \hline\end{array}$$

39. $3x+4a)81x^4-256a^4(27x^3-36ax^2+48a^2x-64a^3$

$$\begin{array}{l}81x^4+108ax^3\\ \hline -108ax^3-256a^4\\ -108ax^3-144a^2x^2\\ \hline 144a^2x^2-256a^4\\ 144a^2x^2+192a^3x\\ \hline -192a^3x-256a^4\\ -192a^3x-256a^4\\ \hline\end{array}$$

40. $a^2-b^2)2a^3+3a^2b-2ab^2-3b^3(2a+3b$

$$\begin{array}{l}2a^3-2ab^2\\ \hline 3a^2b-3b^3\\ 3a^2b-3b^3\\ \hline\end{array}$$

41. $x^2-a^2)x^3+2ax^2-a^2x-2a^3(x+2a$

$$\begin{array}{l}x^3-a^2x\\ \hline 2ax^2-2a^3\\ 2ax^2-2a^3\\ \hline\end{array}$$

42. $a^2+3b^2)a^4-a^2b^2-12b^4(a^2-4b^2$

$$\begin{array}{l}a^4+3a^2b^2\\ \hline -4a^2b^2-12b^4\\ -4a^2b^2-12b^4\\ \hline\end{array}$$

43. $$x^2+3x+y)x^4-9x^2-6xy-y^2(x^2-3x-y$$
$$\underline{x^4+3x^3+x^2y}$$
$$-3x^3-x^2y-9x^2-6xy-y^2$$
$$\underline{-3x^3-9x^2-3xy}$$
$$-x^2y-3xy-y^2$$
$$\underline{-x^2y-3xy-y^2}$$

44. $$x^2-3xy+2y^2)x^4-6x^3y+9x^2y^2-4y^4(x^2-3xy-2y^2$$
$$\underline{x^4-3x^3y+2x^2y^2}$$
$$-3x^3y+7x^2y^2-4y^4$$
$$\underline{-3x^3y+9x^2y^2-6xy^3}$$
$$-2x^2y^2+6xy^3-4y^4$$
$$\underline{-2x^2y^2+6xy^3-4y^4}$$

45. $$x-3y)x^4-81y^4(x^3+3x^2y+9xy^2+27y^3$$
$$\underline{x^4-3x^3y}$$
$$3x^3y-81y^4$$
$$\underline{3x^3y-9x^2y^2}$$
$$9x^2y^2-81y^4$$
$$\underline{9x^2y^2-27xy^3}$$
$$27xy^3-81y^4$$
$$\underline{27xy^3-81y^4}$$

46. $$a-2b)a^4-16b^4(a^3+2a^2b+4ab^2+8b^3$$
$$\underline{a^4-2a^3b}$$
$$2a^3b-16b^4$$
$$\underline{2a^3b-4a^2b^2}$$
$$4a^2b^2-16b^4$$
$$\underline{4a^2b^2-8ab^3}$$
$$8ab^3-16b^4$$
$$\underline{8ab^3-16b^4}$$

47. $3a+2b)81a^4-16b^4(27a^3-18a^2b+12ab^2-8b^3$

$$\begin{array}{l}
81a^4+54a^3b \\
\hline
\quad -54a^3b-16b^4 \\
\quad -54a^3b-36a^2b^2 \\
\hline
\qquad\quad 36a^2b^2-16b^4 \\
\qquad\quad 36a^2b^2+24ab^3 \\
\hline
\qquad\qquad\quad -24ab^3-16b^4 \\
\qquad\qquad\quad -24ab^3-16b^4 \\
\hline
\end{array}$$

48. $2x+3y)\,16x^4-81y^4(8x^3-12x^2y+18xy^2-27y^3$

$$\begin{array}{l}
16x^4+24x^3y \\
\hline
\quad -24x^3y-81y^4 \\
\quad -24x^3y-36x^2y^2 \\
\hline
\qquad\quad 36x^2y^2-81y^4 \\
\qquad\quad 36x^2y^2+54xy^3 \\
\hline
\qquad\qquad\quad -54xy^3-81y^4 \\
\qquad\qquad\quad -54xy^3-81y^4 \\
\hline
\end{array}$$

49. $a+2b+3c)\,3a^2+8ab+4b^2+10ac+8bc+3c^2(3a+2b+c$

$$\begin{array}{l}
3a^2+6ab+9ac \\
\hline
\quad 2ab+4b^2+ac+8bc \\
\quad 2ab+4b^2+6bc \\
\hline
\qquad\quad ac+2bc+3c^2 \\
\qquad\quad ac+2bc+3c^2 \\
\hline
\end{array}$$

50. $a^2+2ax+4x^2)\,a^4+4a^2x^2+16x^4(a^2-2ax+4x^2$

$$\begin{array}{l}
a^4+2a^3x\ +4a^2x^2 \\
\hline
\quad -2a^3x\ +16x^4 \\
\quad -2a^3x\ -4a^2x^2-8ax^3 \\
\hline
\qquad\quad 4a^2x^2+8ax^3+16x^4 \\
\qquad\quad 4a^2x^2+8ax^3+16x^4 \\
\hline
\end{array}$$

51. $x^2 - xy + y^2)\,x^4 + x^2y^2 + y^4\,(x^2 + xy + y^2$

$$\underline{x^4 - x^3y + x^2y^2}$$
$$x^3y + y^4$$
$$\underline{x^3y - x^2y^2 + xy^3}$$
$$x^2y^2 - xy^3 + y^4$$
$$\underline{x^2y^2 - xy^3 + y^4}$$

52. $16x^2 + 4xy + y^2)\,256x^4 + 16x^2y^2 + y^4\,(16x^2 - 4xy + y^2$

$$\underline{256x^4 + 64x^3y + 16x^2y^2}$$
$$-64x^3y + y^4$$
$$\underline{-64x^3y - 16x^2y^2 - 4xy^3}$$
$$16x^2y^2 + 4xy^3 + y^4$$
$$\underline{16x^2y^2 + 4xy^3 + y^4}$$

53. $x^3 + x - y)\,x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3\,(x^2 + xy - y^2$

$$\underline{x^5 + x^3 - x^2y}$$
$$x^4y - x^3y^2 + x^2y - 2xy^2 + y^3$$
$$\underline{x^4y + x^2y - xy^2}$$
$$-x^3y^2 - xy^2 + y^3$$
$$\underline{-x^3y^2 - xy^2 + y^3}$$

54. $x - a)\,ax^3 + 3a^2x^2 - 2a^3x - 2a^4\,(ax^2 + 4a^2x + 2a^3$

$$\underline{ax^3 - a^2x^2}$$
$$4a^2x^2 - 2a^3x$$
$$\underline{4a^2x^2 - 4a^3x}$$
$$2a^3x - 2a^4$$
$$\underline{2a^3x - 2a^4}$$

55. $a + x)\,a^2 - x^2\,(a - x$

$$\underline{a^2 + ax}$$
$$-ax - x^2$$
$$\underline{-ax - x^2}$$

56. $2x + 3y + z)\,2x^2 + xy - 3y^2 - 4yz - xz - z^2\,(x - y - z$

$$\underline{2x^2 + 3xy + xz}$$
$$-2xy - 3y^2 - 4yz - 2xz - z^2$$
$$\underline{-2xy - 3y^2 - yz}$$
$$-2xz - 3yz - z^2$$
$$\underline{-2xz - 3yz - z^2}$$

57. $1 + 5x + x^2)\,2 + 9x + 14x^3 + 3x^4\,(2 - x + 3x^2$

$$\begin{array}{l} 2 + 10x + 2x^2 \\ \hline -x - 2x^2 + 14x^3 \\ -x - 5x^2 - x^3 \\ \hline 3x^2 + 15x^3 + 3x^4 \\ 3x^2 + 15x^3 + 3x^4 \\ \hline \end{array}$$

58. $3 - 5x + 7x^2)\,12 - 38x + 82x^2 - 112x^3 + 106x^4 - 70x^5\,(4 - 6x + 8x^2 - 10x^3$

$$\begin{array}{l} 12 - 20x + 28x^2 \\ \hline -18x + 54x^2 - 112x^3 \\ -18x + 30x^2 - 42x^3 \\ \hline 24x^2 - 70x^3 + 106x^4 \\ 24x^2 - 40x^3 + 56x^4 \\ \hline -30x^3 + 50x^4 - 70x^5 \\ -30x^3 + 50x^4 - 70x^5 \\ \hline \end{array}$$

59. $x^4 - x^3y + x^2y^2 - xy^3 + y^4)\,x^5 + y^5\,(x + y$

$$\begin{array}{l} x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 \\ \hline x^4y - x^3y^2 + x^2y^3 - xy^4 + y^5 \\ x^4y - x^3y^2 + x^2y^3 - xy^4 + y^5 \\ \hline \end{array}$$

60. $ax + xy + ab + by)\,a^2x^2 - a^2b^2 - x^2y^2 + b^2y^2\,(ax - xy - ab + by$

$$\begin{array}{l} a^2x^2 + ax^2y + a^2bx + abxy \\ \hline -ax^2y - a^2bx - abxy - a^2b^2 - x^2y^2 + b^2y^2 \\ -ax^2y - x^2y^2 - abxy - bxy^2 \\ \hline -a^2bx + bxy^2 - a^2b^2 + b^2y^2 \\ -a^2bx - abxy - a^2b^2 - ab^2y \\ \hline abxy + bxy^2 + ab^2y + b^2y^2 \\ abxy + bxy^2 + ab^2y + b^2y^2 \\ \hline \end{array}$$

61. $ax + by)\, abx^2 + a^2xy + b^2xy + aby^2\,(bx + ay$

$$\underline{abx^2 + b^2xy}$$

$$a^2xy + aby^2$$

$$\underline{a^2xy + aby^2}$$

62. $x^2 + ax + b^2)\, x^4 + 2b^2x^2 - a^2x^2 + b^4\,(x^2 - ax + b^2$

$$\underline{x^4 + ax^3 + b^2x^2}$$

$$-ax^3 + b^2x^2 - a^2x^2$$

$$\underline{-ax^3 - a^2x^2 - ab^2x}$$

$$b^2x^2 + ab^2x + b^4$$

$$\underline{b^2x^2 + ab^2x + b^4}$$

XV.

1. $x^2 - ax + c)\, x^4 - (a^2 - b - c)x^2 - (b - c)ax + bc\,(x^2 + ax + b$

$$\underline{x^4 - ax^3 + cx^2}$$

$$ax^3 - (a^2 - b)x^2 - (ab - ac)x + bc$$

$$\underline{ax^3 - a^2x^2 + acx}$$

$$bx^2 - abx + bc$$

$$\underline{bx^2 - abx + bc}$$

2. $y - n)\, y^3 - (l + m + n)y^2 + (lm + ln + mn)y - lmn\,(y^2 - (l + m)y + lm$

$$\underline{y^3 - ny^2}$$

$$-(l + m)y^2 + (lm + ln + mn)y - lmn$$

$$\underline{-(l + m)y^2 + (ln + mn)y}$$

$$lmy - lmn$$

$$\underline{lmy - lmn}$$

3. $x^3 - mx^2 + nx + r)x^5 - (m-c)x^4 + (n - cm + d)x^3 + (r + cn - dm)x^2 + (cr + dn)x + dr(x^2 + cx + d$

$$\begin{array}{l} x^5 - mx^4 + nx^3 + rx^2 \\ \hline cx^4 + (d - cm)x^3 + (cn - dm)x^2 \\ cx^4 - cmx^3 + cnx^2 + crx \\ \hline dx^3 - dmx^2 + dnx + dr \\ dx^3 - dmx^2 + dnx + dr \\ \hline \end{array}$$

4. $x^2 + 5x - 4)x^4 + (5+a)x^3 - (4 - 5a + b)x^2 - (4a + 5b)x + 4b(x^2 + ax - b$

$$\begin{array}{l} x^4 + 5x^3 - 4x^2 \\ \hline ax^3 + (5a - b)x^2 - (4a + 5b)x \\ ax^3 + 5ax^2 - 4ax \\ \hline -bx^2 - 5bx + 4b \\ -bx^2 - 5bx + 4b \\ \hline \end{array}$$

5. $x^2 - (a+c)x + ac)x^4 - (a+b+c+d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - (abc + abd + acd + bcd)x + abcd(x^2 - (b+d)x + bd$

$$\begin{array}{l} x^4 - (a+c)x^3 + acx^2 \\ \hline -(b+d)x^3 + (ab + ad + bc + bd + cd)x^2 \\ -(b+d)x^3 + (ab + ad + bc + cd)x^2 - (abc + acd)x \\ \hline bdx^2 - (abd + bcd)x + abcd \\ bdx^2 - (abd + bcd)x + abcd \\ \hline \end{array}$$

XVIII.

1. $(x^2 - ax) - (bx - ab) = x(x-a) - b(x-a) = (x-b)(x-a)$.

2. $(ab + ax) - (bx + x^2) = a(b+x) - x(b+x) = (a-x)(b+x)$.

3. $(bc + by) - (cy + y^2) = b(c+y) - y(c+y) = (b-y)(c+y)$.

4. $(bm + mn) + (ab + an) = m(b+n) + a(b+n) = (a+m)(b+n)$.

5. $(abx^2 - axy) + (bxy - y^2) = ax(bx - y) + y(bx - y) = (ax + y)(bx - y).$

6. $(abx - aby) + (cdx - cdy) = ab(x - y) + cd(x - y) = (ab + cd)(x - y).$

7. $(cdx^2 + dmxy) - (cnxy + mny^2) = dx(cx + my) - ny(cx + my)$
$= (cx + my)(dx - ny).$

8. $(abcx - b^2dx) - (acdy - bd^2y) = bx(ac - bd) - dy(ac - bd)$
$= (ac - bd)(bx - dy).$

XXVI.

15. $(x^2 - 2xy + y^2) - z^2 = (x - y)^2 - z^2 = (x - y + z)(x - y - z).$

17. $(a^2 - 2ac + c^2) - (b^2 + 2bd + d^2) = (a - c)^2 - (b + d)^2$, etc.

18. $a^2 - (b^2 - 2bc + c^2) = a^2 - (b - c)^2$, etc.

19. $(x^2 + 2xy + y^2) - z^2 = (x + y)^2 - z^2$, etc.

20. $(a^2 - 2ab + b^2) - (m^2 - 2mn + n^2) = (a - b)^2 - (m - n)^2$, etc.

23. $1 - (a^2 - 2ab + b^2) = 1 - (a - b)^2$, etc.

24. $1 - (x^2 - 2xy + y^2) = 1 - (x - y)^2$, etc.

25. $x^2 - (y^2 + 2yz + z^2) = x^2 - (y + z)^2$, etc.

26. $a^2 - (4b^2 - 12bc + 9c^2) = a^2 - (2b - 3c)^2$, etc.

29. $(a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2$, etc.

30. $(a^2 - 2ac + c^2) - (b^2 - 2bd + d^2) = (a - c)^2 - (b - d)^2$, etc.

31. $3ax(a^2x^2 - 9) = 3ax(ax + 3)(ax - 3).$

XXVII.

11. $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$

12. $x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1).$

13. $a^6 - 64 = a^6 - 2^6 = (a^3 + 2^3)(a^3 - 2^3)$
$= (a+2)(a^2 - 2a + 4)(a-2)(a^2 + 2a + 4)$.

14. $729 - y^6 = 3^6 - y^6 = (3^3 + y^3)(3^3 - y^3)$
$= (3+y)(9 - 3y + y^2)(3-y)(9 + 3y + y^2)$

XXX.

1. $6+4-5-3=2$.

2. $6+3-5-4=0$.

3. $18+12-5-8=17$.

4. $3 \times 11 - 2 \times 1 = 33 - 2 = 31$.

5. $10 \times 2 = 20$.

6. $12 + 3 \times 7 = 12 + 21 = 33$.

7. $(12+3)(4+3) = 15 \times 7 = 105$.

8. $12+12+3=27$.

9. $(25+3) \div (6-4) = 28 \div 2 = 14$.

10. $6 \times 5 \times 4 = 120$.

11. $6 \times 5 \times 7 = 210$.

12. $6 \times 3 \times 9 \times 9 = 1458$.

13. $6 \times 5 \times 1 \times 1 = 30$.

14. $\sqrt{5 \times 5} = 5$.

15. $\sqrt{y^2} = y = 3$.

16. $(\sqrt{x})^2 = (2)^2 = 4$.

17. $(\sqrt{x} + b)^2 = (2+5)^2 = 49$.

18. $\sqrt{5 \times 5 \times 4} = \sqrt{100} = 10$.

19. $\sqrt{2 \times 6 \times 4 \times 3} = \sqrt{144} = 12.$

20. $(36 + 25 + 3) \div (4 + 9 + 3) = 64 \div 16 = 4.$

21. $18 + (8 - 3)^2 = 18 + 25 = 43.$

22. $\{6 - (5 - 3)\}\{6 - (4 - 3)\} = 4 \times 5 = 20.$

23. $(6 - 5 - 3)^2 + (6 - 4 + 3)^2 = 4 + 25 = 29.$

24. $3 \times 8^3 + 4 \times 10^4 = 1536 + 40000$, etc.

25. $3 \times 1^2 + 7^2 = 3 + 49 = 52.$

XXXI.

1. $3 \times 3 \times 2 \times 1 - 3^3 + 2^3 + 1^3 = 18 - 27 + 8 + 1 = 0.$

2. $3^3 + 2^3 - 5^3 + 3 \times 3 \times 2 \times 5 = 27 + 8 - 125 + 90 = 0.$

3. $(a^2 + 2ac + c^2) - (a^2 + c^2) = 2ac.$

4. $x^2 + y^2 - (x^2 - 2xy + y^2) = 2xy.$

5. $(a^2 + 2ab + b^2)x - (a^2 + 2abx + b^2x^2) = (a^2 + b^2)x - a^2 - b^2x^2.$

6. Multiply $2x - m$ by $x + 2m$; the result is $2x^2 + 3mx - 2m^2$.
   Multiply $2x + n$ by $x - 2n$; the result is $2x^2 - 3nx - 2n^2$.
   Then multiply the two results together.

7. $$\begin{array}{l} ar + b\big)\,acr^3 + (bc + ad)r^2 + (bd + ae)r + be\,\big(cr^2 + dr + e \\ \qquad\;\; \underline{acr^3 + bcr^2} \\ \qquad\qquad\quad adr^2 + (bd + ae)r \\ \qquad\qquad\quad \underline{adr^2 + bdr} \\ \qquad\qquad\qquad\quad aer + be \\ \qquad\qquad\qquad\quad \underline{aer + be} \end{array}$$

The divisor $= 10 + 1$, or 11; the dividend $= 1000 + 200 + 20 + 1$, or 1221; the quotient $= 100 + 10 + 1$, or 111; and $1221 \div 11 = 111$.

8. Multiply $a+b+c$ by $a+b-c$; the result is $(a+b)^2-c^2$, or $a^2+2ab+b^2-c^2$.

Multiply $c+b-a$ by $c+a-b$; the result is $c^2-(a-b)^2$, or $c^2-a^2+2ab-b^2$.

Then $(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)=(2ab)^2-(a^2+b^2-c^2)^2$
$$=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2$$
$$=-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2.$$

9. $(a+d)(d+c)-(c+d)(d+a)-(a+c)(d-d)$
$$=ad+ac+d^2+cd-cd-ac-d^2-ad-0=0.$$

10. $0+(4-16)+\{16-(0+6)\}+\{12-(0+6)\}^2$
$$=-12+10+6^2=-2+36=34.$$

11. $\{4-3+2-1\}\{4+3-2-1\}=2\times4=8$;
and $16-9-4+1+2\times2=8$.

18. $(ac-ad-bc+bd)-(bc-bd-ac+ad)=2ac-2ad-2bc+2bd.$

19. $ab+ay+bx+xy+x-y+ax-by+ax+ay=$ etc.

20. $(x^2+3x+2)-(x^2-3x+2)=6x$, etc.

21. $ax-by+x-y+x^2-xy+ab-ay-bx+xy$, etc.

22. $(6x^2+20x+16)-(6x^2-28x+16)=48x$, etc.

23. $2mx-3ny+x+y+4mx+4nx-4my-4ny+mx+ny=$ etc.

24. $x^2+y^2+z^2+2xy+2xz+2yz+x^2+y^2+z^2$
$$=(x^2+2xy+y^2)+(y^2+2yz+z^2)+(x^2+2xz+z^2)$$
$$=(x+y)^2+(y+z)^2+(x+z)^2.$$

25. $4a^2+6ac-4ab+6ab+9bc-6b^2=4a^2+6ac+2ab+9bc-6b^2.$

26. $\dfrac{ab-cd}{cd+e}=\dfrac{63-15}{15+1}=\dfrac{48}{16}=3.$

$(bc-ad)(bd-ce)=(35-27)(21-5)=8\times16=128.$

$\dfrac{b^2-c^2}{c+d}=\dfrac{49-25}{5+3}=\dfrac{24}{8}=3.$

$d^e-c^d=3^5-5^3=243-125=118.$

27. $0-0+2^3+1^3=8+1=9$.

28. $48+4-8=44$.

29. $(1-2-3)^2+(2-1-3)^2+(3-1-2)^2=(-4)^2+(-2)^2+0=16+4$
$=20$.

30. $(1+2-4)^2+(1-2+4)^2+(2+4-1)^2=(-1)^2+(3)^2+(5)^2$
$=1+9+25=35$.

31. $(-1+2)^2+(2-3)^2+(-1-3)^2=(1)^2+(-1)^2+(-4)^2=1+1+16$
$=18$.

32. Let x and y be the numbers; then $(x^2-y^2)\div(x+y)=x-y$.

33. Let x and y be the numbers; then $(x+y)(x-y)=x^2-y^2$.

34. Let x and $x+1$ be the integers; then $(x+x+1)^2=4x^2+4x+1$
$=4x(x+1)+1$.

35. Let x and $x+2$ be the two even numbers; then $x+1$ is the odd number between them; and $(x+x+2)^2=4x^2+8x+4=4(x+1)^2$.

36. Let x and y be the parts; then $x+y=2$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=2(x-y)$.

37. Let x and y be the parts; then $x+y=50$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=50(x-y)$.

38. Let x and y be the parts; then $x+y=n$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=n(x-y)$.

39. Let x and $x+1$ be the numbers; then $x(x+1)+x^2+(x+1)^2$
$=x^2+x+x^2+x^2+2x+1=3x^2+3x+1=(x+1)^3-x^3$.

40. Let $x-1$, x, $x+1$ be the three numbers; then $(x-1)^3+x^3+(x+1)^3$
$=x^3-3x^2+3x-1+x^3+x^3+3x^2+3x+1=3x^3+6x=3x(x^2+2)$.

XXXII.

1. $7x-5x=11-5$; $2x=6$; $x=3$.

2. $12x-8x=15-7$; $4x=8$; $x=2$.

3. $236x-97x=564-425$; $139x=139$; $x=1$.

4. $5x-3x=7+7$; $2x=14$; $x=7$.

5. $12x-8x=9-1$; $4x=8$; $x=2$.

6. $124x-112x=43-19$; $12x=24$; $x=2$.

7. $5x-2x=27-18$; $3x=9$; $x=3$.

8. $12x-7x=145-125$; $5x=20$; $x=4$.

9. $14x-8x=80-26$; $6x=54$; $x=9$.

10. $-3x-x=-83-133$; $-4x=-216$; $4x=216$; $x=54$.

11. $-3x-5x=-3-13$; $-8x=-16$; $8x=16$; $x=2$.

12. $9x-12x=100-127$; $-3x=-27$; $3x=27$; $x=9$.

13. $-5x+4x=6-15$; $-x=-9$; $x=9$.

14. $3x-7x=6+22$; $-4x=28$; $4x=-28$; $x=-7$.

15. $4x-12x=-16-8$; $-8x=-24$; $8x=24$; $x=3$.

16. $5x-3x+7=4x-6x+35$; $4x=28$; $x=7$.

17. $6x-18+8x+15x-21=10x-4-16x+35$; $35x=70$; $x=2$.

18. $9x-15x+18+30=0$; $-6x=-48$; $x=8$.

19. $12x-45x-15+42-48x+783=0$; $-81x=-810$; $x=10$.

20. $x-28x+77=14x-70-152+19x-61$; $-60x=-360$; $x=6$.

21. $x^2+4x-21=x^2-20x+75$; $24x=96$; $x=4$.

22. $x^2+4x-96=x^2-5x-6$; $9x=90$; $x=10$.

23. $9x-x^2-14+x^2-2x-15-2x+2+12=0$; $5x=15$; $x=3$.

24. $2x^2+3x-35=36-17x+2x^2+229$; $20x=300$; $x=15$.

25. $21-32x+12x^2=12x^2-17x+6$; $-15x=-15$; $x=1$.

26. $14-x-5x^2+5x+30+20-29x+5x^2=45x-76$; $-70x=-140$; $x=2$.

27. $x^2+10x+25-16+8x-x^2=21x$; $-3x=-9$; $x=3$.

28. $5x^2-20x+20+7x^2-42x+63=12x^2-85x+133+42$; $23x=92$; $x=4$.

29. $9x^2-102x+289+16x^2-200x+625-25x^2+290x-841=1$; $-12x=-72$; $x=6$.

30. $x^2-4x-45+x^2+2x-80=2x^2-11x-21-113$; $9x=-9$; $x=-1$.

XXXIII.

1. Let x be the number; then $2x+14=154$; $x=70$.

2. Let x be the number; then $4x+16=188$; $x=43$.

3. Let x be the number; then $x+46=3x$; $x=23$.

4. Let x be the smaller number; then $3x$ is the greater number; and $16-x=30-3x$; $2x=14$; $x=7$; $3x=21$.

5. Let x be the first part; then $x-10$ is the second; $x-18$ the third; $x-24$ the fourth; and $x+x-10+x-18+x-24=92$; $x=36$, etc.

6. Let x be the greater number; then $20-x$ is the smaller number; and $3(20-x)+5x=84$; $x=12$, etc.

7. Let x be the father's age in years; then $80-x$ is the son's age; and $2(80-x)=x+10$; $x=50$, etc.

8. Let x be the age of the eldest; then $x-20$ is the age of the youngest; and $x=3(x-20)$; $x=30$, etc.

9. Let x be the sum in pounds; then $x+24-80=80-x$; $x=68$.

10. Let x be the price of a yard of cloth in shillings; then $2x$ is the price of a yard of silk; and $30x+80x=66\times 20$; $x=12$, etc.

11. Let x be the number; then $2x+24-80=100-x$; $x=52$.

12. Let x be A's share in pounds; then B's share is $280-x$; C's, $260-x$; D's, $220-x$; and $x+280-x+260-x+220-x=500$; $x=130$, etc.

13. Let x be the number of children; then there are $2x$ women and $4x$ men; and $x+2x+4x=266$; $x=38$, etc.

14. Let x be B's share in pounds; then A's share is $x-100$, and C's is $x+270$; and $x+x-100+x+270=1520$; $x=450$, etc.

15. Let x be the greater; then $x-8$ is the less; and $4(x-8)=2x+10$; $x=21$; $x-8=13$.

16. Let x be what each had at first in pounds; then $3(x+5)=11(x-5)$; $x=$ £8, 15s.

17. Let x be A's age; then $x-58$ is B's age; and $x-60=50-(x-58)$; $x=84$, etc.

18. Let x be A's age; then $x-34$ is B's age; and $x-50=40-(x-34)$; $x=62$, etc.

19. Let x be the share of a daughter; then $2x$ is the share of a son; and $3x+4x+500$ is the share of the wife; then $3x+4x+3x+4x+500=7500$; $x=500$, etc.

20. Let x be the number of gallons in the vessel at first; then $x+42=7x$; $x=7$; and $42+7=49$, the number of gallons that the vessel held.

21. Let x be the number of pounds A has; then B has $x+10$; C has $x+x+10$; and $x+x+10+x+x+10=76$; $x=14$, etc.

22. Let x be the greater; then $x-14$ is the less; and $x+x-14=48$; $x=31$, etc.

23. Let x be the number of pounds won by A; then $72+x=3(52-x)$; $x=21$.

24. Let x be one of the parts; then $84-x$ is the other; and $3x=4(84-x)$; $x=48$.

25. Let x be one of the parts; then $90-x$ is the other; and $4x=5(90-x)$; $x=50$.

26. Let x be the greater part; then $60-x$ is the less; and $x=60-x+24$; $x=42$.

27. Let x be the greater part; then $84-x$ is the less; and $x-36=84-x$; $x=60$.

28. Let x be one of the parts; then $20-x$ is the other; and $3x+5(20-x)=84$; $x=8$.

29. Let x be B's age; then $2x$ is A's age; and $2x-22=3(x-22)$; $x=44$; $2x=88$.

30. Let x be the number of years; then $30+x=2(6+x)$; $x=18$.

31. Let x be B's age; then $2x$ is A's age; and $2x-20=3(x-20)$; $x=40$.

32. Let x be B's age; then $3x$ is A's age; and $3x+19=2(x+19)$; $x=19$.

33. Let x be the number of years; then $50+x=42+3x$; $x=4$.

34. Let x be the number of guineas; then $x+48$ is the number of half-crowns; and, expressing all the quantities as sixpences, $42x+5(x+48)=100\times 40$; $47x=3760$; $x=80$.

35. Let x be the number of shillings; then $41-x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+5(41-x)=74\times 2$; $-3x=-57$; $x=19$.

36. Let x be the number of shillings; then $300-x$ is the number of fourpenny pieces; and, expressing all the quantities as fourpenny pieces, $3x+300-x=700$; $x=200$.

37. Let x be the number of moidores; then $x+3$ is the number of sovereigns; and $27x+20(x+3)=50\times 20$; $47x=940$; $x=20$.

38. Let x be the number of shillings; then $6x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+30x=1696$; $x=53$.

39. Let x be the number of sovereigns; then $2x$ is the number of shillings, and $3x$ the number of pence; and, expressing all the quantities as pence, $240x+24x+3x=1335$; $x=5$.

XXXV.

1. $a^2-b^2=(a-b)(a+b)$; $a^3-b^3=(a-b)(a^2+ab+b^2)$.

2. $a^4-b^4=(a^2-b^2)(a^2+b^2)$.

3. $a^2-x^2=(a+x)(a-x)$; $(a-x)^2=(a-x)(a-x)$.

4. $a^3+x^3=(a+x)(a^2-ax+x^2)$; $(a+x)^3=(a+x)(a+x)(a+x)$.

5. $9x^2-1=(3x+1)(3x-1)$; $(3x+1)^2=(3x+1)(3x+1)$.

6. $1-25a^2=(1+5a)(1-5a)$; $(1-5a)^2=(1-5a)(1-5a)$.

7. $x^2-y^2=(x+y)(x-y)$; $(x+y)^2=(x+y)(x+y)$;
$x^2+3xy+2y^2=(x+y)(x+2y)$.

8. $x^2-y^2=(x+y)(x-y)$; $x^3-y^3=(x-y)(x^2+xy+y^2)$;
$x^2-7xy+6y^2=(x-y)(x-6y)$.

9. $x^2-1=(x-1)(x+1)$; $x^3-1=(x-1)(x^2+x+1)$;
$x^2+x-2=(x-1)(x+2)$.

10. $1-a^2=(1-a)(1+a)$; $1+a^3=(1+a)(1-a+a^2)$;
$a^2+5a+4=(1+a)(4+a)$.

XXXVI.

1.
```
6906)10359(1
     6906
     ----
     3453)6906(2
          6906
          ----
```
∴ H. C. F. is 3453.

2.
```
1908)2736(1
     1908
     ----
      828)1908(2
          1656
          ----
           252)828(3
               756
               ---
                72)252(3
                   216
                   ---
                    36)72(2
                       72
                       --
```
∴ H. C. F. is 36.

3.
```
49608)169416(3
      148824
      ------
       20592)49608(2
             41184
             -----
              8424)20592(2
                   16848
                   -----
                    3744)8424(2
                         7488
                         ----
                          936)3744(4
                              3744
                              ----
```
∴ H. C. F. is 936.

4. 40115)126025(3
120345
5680)40115(7
39760
355)5680(16
355
2130
2130

∴ H. C. F. is 355.

5. 1581227)16758766(10
15812270
946496)1581227(1
946496
634731)946496(1
634731
311765)634731(2
623530
11201)311765(27
22402
87745
78407
9338)11201(1
9338
1863)9338(5
9315
23)1863(81
184
23
23

∴ H. C. F. is 23.

6.
$$
\begin{array}{l}
35175)236845(6 \\
\quad\underline{211050} \\
\quad 25795)35175(1 \\
\qquad\underline{25795} \\
\qquad 9380)25795(2 \\
\qquad\quad\underline{18760} \\
\qquad\quad 7035)9380(1 \\
\qquad\qquad\underline{7035} \\
\qquad\qquad 2345)7035(3 \\
\qquad\qquad\quad\underline{7035}
\end{array}
$$

$\therefore$ H. C. F. is 2345.

XXXVII.

1.
$$
\begin{array}{l}
x^2+7x+12)x^2+9x+20(1 \\
\qquad\qquad\quad\underline{x^2+7x+12} \\
\qquad\qquad\qquad 2x+8
\end{array}
$$

Divide by 2;
$$
\begin{array}{l}
x+4)x^2+7x+12(x+3 \\
\qquad\underline{x^2+4x} \\
\qquad\quad 3x+12 \\
\qquad\quad\underline{3x+12}
\end{array}
$$

2.
$$
\begin{array}{l}
x^2+12x+20)x^2+14x+40(1 \\
\qquad\qquad\quad\underline{x^2+12x+20} \\
\qquad\qquad\qquad 2x+20
\end{array}
$$

Divide by 2;
$$
\begin{array}{l}
x+10)x^2+12x+20(x+2 \\
\qquad\underline{x^2+10x} \\
\qquad\quad 2x+20 \\
\qquad\quad\underline{2x+20}
\end{array}
$$

3. $x^2-13x+42)\,x^2-17x+70\,(1$
$$\begin{array}{r}x^2-13x+42\\ \hline -4x+28\end{array}$$

Divide by 4 and change signs ; $x-7)\,x^2-13x+42\,(x-6$
$$\begin{array}{r}x^2-7x\\ \hline -6x+42\\ -6x+42\\ \hline\end{array}$$

4. $x^2+5x-84)\,x^2+21x+108\,(1$
$$\begin{array}{r}x^2+\ 5x-\ 84\\ \hline 16x+192\end{array}$$

Divide by 16 ; $x+12)\,x^2+5x-84\,(x-7$
$$\begin{array}{r}x^2+12x\\ \hline -7x-84\\ -7x-84\\ \hline\end{array}$$

5. $x^2+x-12)\,x^2-2x-3\,(1$
$$\begin{array}{r}x^2+x-12\\ \hline -3x+9\end{array}$$

Change signs and divide by 3 ; $x-3)\,x^2+x-12\,(x+4$
$$\begin{array}{r}x^2-3x\\ \hline 4x-12\\ 4x-12\\ \hline\end{array}$$

6. $x^2+5xy+6y^2)\,x^2+6xy+9y^2\,(1$
$$\begin{array}{r}x^2+5xy+6y^2\\ \hline xy+3y^2\end{array}$$

Divide by y ; $x+3y)\,x^2+5xy+6y^2\,(x+2y$
$$\begin{array}{r}x^2+3xy\\ \hline 2xy+6y^2\\ 2xy+6y^2\\ \hline\end{array}$$

7. $x^2-6xy+8y^2)x^2-8xy+16y^2(1$

$$x^2-6xy+8y^2$$

$$-2xy+8y^2$$

Divide by $2y$ and change signs ; $x-4y)x^2-6xy+8y^2(x-2y$

$$x^2-4xy$$

$$-2xy+8y^2$$

$$-2xy+8y^2$$

8. $x^2-13xy-30y^2)x^2-18xy+45y^2(1$

$$x^2-13xy-30y^2$$

$$-5xy+75y^2$$

Divide by $5y$ and change signs ; $x-15y)x^2-13xy-30y^2(x+2y$

$$x^2-15xy$$

$$2xy-30y^2$$

$$2xy-30y^2$$

9. $x^2-2xy+y^2)x^3-y^3(x+2y$

$$x^3-2x^2y+xy^2$$

$$2x^2y-xy^2-y^3$$

$$2x^2y-4xy^2+2y^3$$

$$3xy^2-3y^3$$

Divide by $3y^2$; $x-y)x^2-2xy+y^2(x-y.$

10. $x^3+y^3)x^3+3x^2y+3xy^2+y^3(1$

$$x^3 \qquad\qquad +y^3$$

$$3x^2y+3xy^2$$

Divide by $3xy$; $x+y)x^3+y^3(x^2-xy+y^2.$

11. $x^2-2xy+y^2)\,x^4-y^4\,(x^2+2xy+3y^2$

$$\underline{x^4-2x^3y+x^2y^2}$$
$$2x^3y-x^2y^2-y^4$$
$$\underline{2x^3y-4x^2y^2+2xy^3}$$
$$3x^2y^2-2xy^3-y^4$$
$$\underline{3x^2y^2-6xy^3+3y^4}$$
$$4xy^3-4y^4$$

Divide by $4y^3$; $x-y)\,x^2-2xy+y^2\,(x-y$.

12. $x^3+y^3)\,x^5+y^5\,(x^2$

$$\underline{x^5+x^2y^3}$$
$$-x^2y^3+y^5$$

Divide by $-y^3$; $x^2-y^2)\,x^3+y^3\,(x$

$$\underline{x^3-xy^2}$$
$$xy^2+y^3$$

Divide by y^2; $x+y)\,x^2-y^2\,(x-y$.

13. $x^2+2xy+y^2)\,x^4-y^4\,(x^2-2xy+3y^2$

$$\underline{x^4+2x^3y+x^2y^2}$$
$$-2x^3y-x^2y^2-y^4$$
$$\underline{-2x^3y-4x^2y^2-2xy^3}$$
$$3x^2y^2+2xy^3-y^4$$
$$\underline{3x^2y^2+6xy^3+3y^4}$$
$$-4xy^3-4y^4$$

Divide by $-4y^3$; $x+y)\,x^2+2xy+y^2\,(x+y$.

14. $a^2-b^2+2bc-c^2)\,a^2+2ab+b^2-2ac-2bc+c^2\,(1$

$$\underline{a^2-b^2+2bc-c^2}$$
$$2ab+2b^2-2ac-4bc+2c^2$$

Divide by 2; $ab+b^2-ac-2bc+c^2=(ab-ac)+(b^2-2bc+c^2)$.

$=a(b-c)+(b-c)^2$.

Divide by $b-c$; $a+b-c)\,a^2-b^2+2bc-c^2\,(a-b+c$.

15. **Multiply the second expression by 3 ;**

$$12x^2+7xy+y^2)84x^2+9xy-3y^2(7$$
$$84x^2+49xy+7y^2$$
$$-40xy-10y^2$$

Divide by $-10y$; $4x+y)12x^2+7xy+y^2(3x+y.$

16. **Multiply the second expression by 2 ;**

$$6x^2+xy-y^2)78x^2-44xy+6y^2(13$$
$$78x^2+13xy-13y^2$$
$$-57xy+19y^2$$

Divide by $-19y$; $3x-y)6x^2+xy-y^2(2x+y.$

17. **Multiply the second expression by 3 ;**

$$15x^2-8xy+y^2)120x^2-9xy-3y^2(8$$
$$120x^2-64xy+8y^2$$
$$55xy-11y^2$$

Divide by $11y$; $5x-y)15x^2-8xy+y^2(3x-y.$

18. $x^4+x^3-4x^2+x+1)x^5-5x^3+5x^2-1(x-1$

$$x^5+x^4-4x^3+x^2+x$$
$$-x^4-x^3+4x^2-x-1$$
$$-x^4-x^3+4x^2-x-1$$

19. $x^4+4x^2+16)x^5+x^4-2x^3+17x^2-10x+20(x+1$

$$x^5+4x^3+16x$$
$$x^4-6x^3+17x^2-26x+20$$
$$x^4+4x^2+16$$
$$-6x^3+13x^2-26x+4$$

Change the signs of the remainder, and multiply the divisor by 6;

$$6x^3-13x^2+26x-4)6x^4+24x^2+96(x$$
$$6x^4-13x^3+26x^2-4x$$
$$13x^3-2x^2+4x+96$$

Multiply the remainder by 6, and continue the division;

$$\begin{array}{l} 78x^3 - 12x^2 + 24x + 576(13 \\ \underline{78x^3 - 169x^2 + 338x - 52} \\ \qquad 157x^2 - 314x + 628 \end{array}$$

Divide by 157; $x^2 - 2x + 4)6x^3 - 13x^2 + 26x - 4(6x - 1$

$$\begin{array}{l} 6x^3 - 12x^2 + 24x \\ \hline \quad -x^2 + 2x - 4 \\ \quad -x^2 + 2x - 4 \\ \hline \end{array}$$

20. $x^4 + x^2y^2 + y^4)x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4(1$

$$\begin{array}{l} x^4 \qquad + x^2y^2 \qquad + y^4 \\ \hline 2x^3y + 2x^2y^2 + 2xy^3 \end{array}$$

Divide by $2xy$; $x^2 + xy + y^2)x^2 + x^2y^2 + y^4(x^2 - xy + y^2$.

21. $x^6 - 6x^4 + 9x^2 - 4)x^6 + x^5 - 2x^4 + 3x^2 - x - 2(1$

$$\begin{array}{l} x^6 \quad - 6x^4 + 9x^2 \quad - 4 \\ \hline x^5 + 4x^4 - 6x^2 - x + 2 \end{array}$$

$x^5 + 4x^4 - 6x^2 - x + 2)x^6 - 6x^4 + 9x^2 - 4(x - 4$

$$\begin{array}{l} x^6 + 4x^5 - 6x^3 - x^2 + 2x \\ \hline - 4x^5 - 6x^4 + 6x^3 + 10x^2 - 2x - 4 \\ - 4x^5 - 16x^4 + 24x^2 + 4x - 8 \\ \hline \qquad 10x^4 + 6x^3 - 14x^2 - 6x + 4 \end{array}$$

Divide by 2, and multiply the divisor by 5;

$5x^4 + 3x^3 - 7x^2 - 3x + 2)5x^5 + 20x^4 - 30x^2 - 5x + 10(x$

$$\begin{array}{l} 5x^5 + 3x^4 - 7x^3 - 3x^2 + 2x \\ \hline \qquad 17x^4 + 7x^3 - 27x^2 - 7x + 10 \end{array}$$

Multiply by 5, and continue the division;

$$\begin{array}{l} 85x^4 + 35x^3 - 135x^2 - 35x + 50(17 \\ \underline{85x^4 + 51x^3 - 119x^2 - 51x + 34} \\ \qquad - 16x^3 - 16x^2 + 16x + 16 \end{array}$$

Divide by -16; $x^3+x^2-x-1)5x^4+3x^3-7x^2-3x+2(5x-2$

$$\begin{array}{r}
5x^4+5x^3-5x^2-5x \\
\hline
-2x^3-2x^2+2x+2 \\
-2x^3-2x^2+2x+2 \\
\hline
\end{array}$$

22. Multiply the first expression by 2;

$6a^3+19a^2b+8ab^2-5b^3)30a^4+20a^3b+8a^2b^2+12ab^3-6b^4(5a$

$$\begin{array}{r}
30a^4+95a^3b+40a^2b^2-25ab^3 \\
\hline
-75a^3b-32a^2b^2+37ab^3-6b^4
\end{array}$$

Change the signs, multiply by 2, divide by b, and continue the division;

$6a^3+19a^2b+8ab^2-5b^3)150a^3+64a^2b-74ab^2+12b^3(25$

$$\begin{array}{r}
150a^3+475a^2b+200ab^2-125b^3 \\
\hline
-411a^2b-274ab^2+137b^3
\end{array}$$

Divide by $-137b$; $3a^2+2ab-b^2)6a^3+19a^2b+8ab^2-5b^3(2a+5b$.

23. Multiply the second expression by 5;

$15x^3-14x^2y+24xy^2-7y^3)135x^3+165x^2y-100xy^2+10y^3(9$

$$\begin{array}{r}
135x^3-126x^2y+216xy^2-63y^3 \\
\hline
291x^2y-316xy^2+73y^3
\end{array}$$

Divide by y; and multiply the divisor by 97;

$291x^2-316xy+73y^2)1455x^3-1358x^2y+2328xy^2-679y^3(5x$

$$\begin{array}{r}
1455x^3-1580x^2y+365xy^2 \\
\hline
222x^2y+1963xy^2-679y^3
\end{array}$$

Divide by y; multiply by 97, and continue the division;

$$\begin{array}{r}
21534x^2+190411xy-65863y^2(74 \\
21534x^2-23384xy+\ \ 5402y^2 \\
\hline
213795xy-71265y^2
\end{array}$$

Divide by $71265y$; $3x-y)291x^2-316xy+73y^2(97x-73y$.

24. Multiply the second expression by 7;

$$21x^2 - 83xy - 27x + 22y^2 + 99y)\,84x^2 - 245xy - 42x - 231y^2 + 154y\,(4$$
$$\underline{84x^2 - 332xy - 108x + 88y^2 + 396y}$$
$$87xy + 66x - 319y^2 - 242y$$

The remainder $= 87xy - 319y^2 + 66x - 242y$

$= 29y(3x - 11y) + 22(3x - 11y)$

$= (29y + 22)(3x - 11y)$

Rejecting $29y + 22$, which is clearly not a factor of the divisor;

$$3x - 11y)\,21x^2 - 83xy - 27x + 22y^2 + 99y\,(7x - 2y - 9.$$

25. $3a^3 - 12a^2 - a^2b + 10ab - 2b^2 = (3a^3 - a^2b) - (12a^2 - 10ab + 2b^2)$

$= a^2(3a - b) - 2(6a^2 - 5ab + b^2)$

$= a^2(3a - b) - 2(3a - b)(2a - b)$

$= (3a - b)(a^2 - 4a + 2b)$

Rejecting $a^2 - 4a + 2b$, which is clearly not a factor of the second expression, we find the H. C. F. to be $3a - b$.

26. $60a^2 - 75ax + 15x^2 = 15(4a^2 - 5ax + x^2) = 3 \times 5(a - x)(4a - x)$

Of these four factors, 5 and $4a - x$ are clearly not factors of the first expression: but 3 is a factor of it: divide it by 3, and divide the result by $a - x$.

$$a - x)\,6a^3 - 6a^2x + 2ax^2 - 2x^3\,(6a^2 + 2x^2$$

$\therefore$ H. C. F. is $3(a - x)$.

27. Divide the first expression by x, and multiply the result by 2;

$$6x^2 - x - 2)\,42x^2 - 52x + 16\,(7$$
$$\underline{42x^2 - 7x - 14}$$
$$-45x + 30$$

Divide by -15; $3x - 2)\,6x^2 - x - 2\,(2x + 1.$

28. $3x^3 - 15ax^2 + a^2x - 5a^3)\,6x^4 + 29a^2x^2 + 9a^4\,(2x + 10a$

$$\underline{6x^4 - 30ax^3 + 2a^2x^2 - 10a^3x}$$
$$30ax^3 + 27a^2x^2 + 10a^3x + 9a^4$$
$$\underline{30ax^3 - 150a^2x^2 + 10a^3x - 50a^4}$$
$$177a^2x^2 \qquad + 59a^4$$

Divide by $59a^2$; $3x^2 + a^2)\,3x^3 - 15ax^2 + a^2x - 5a^3\,(x - 5a.$

29. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$

$$x^8 + x^6y^2 + x^2y + y^3 = x^6(x^2 + y^2) + y(x^2 + y^2)$$

$$\therefore x^2 + y^2 \text{ is the H. C. F.}$$

30. Divide the first expression by 2;

$$x^3 + x^2 + 7x + 39)\,x^3 + 5x^2 + 7x + 3\,(1$$
$$\underline{x^3 + x^2 + 7x + 39}$$
$$4x^2 - 36$$

Now $4x^2 - 36 = 4(x^2 - 9) = 4(x + 3)(x - 3)$; and rejecting 4 and $x - 3$, we have

$$x + 3)\,x^3 + x^2 + 7x + 39\,(x^2 - 2x + 13.$$

31. $45a^3x + 3a^2x^2 - 9ax^3 + 6x^4 = x(45a^3 + 3a^2x - 9ax^2 + 6x^3)$

$18a^2x - 8x^3 = 2x(9a^2 - 4x^2)$

Reserving the common factor x;

$$9a^2 - 4x^2)\,45a^3 + 3a^2x - 9ax^2 + 6x^3\,(5a$$
$$\underline{45a^3 - 20ax^2}$$
$$3a^2x + 11ax^2 + 6x^3$$

Divide by x, multiply by 3, and proceed with the division;

$$9a^2 + 33ax + 18x^2\,(1$$
$$\underline{9a^2 - 4x^2}$$
$$33ax + 22x^2$$

Divide by $11x$; $3a + 2x)\,9a^2 - 4x^2\,(3a - 2x$

$$\therefore \text{the H. C. F. is } (3a + 2x)x.$$

XXXVIII.

1. $x^2 + 5x + 6 = (x + 2)(x + 3)$; $x^2 + 7x + 10 = (x + 2)(x + 5)$; $x^2 + 12x + 20 = (x + 2)(x + 10)$.

2. $x^3 + 4x^2 - 5)\,x^3 - 3x + 2\,(1$

$$\underline{x^3 + 4x^2 - 5}$$
$$-4x^2 - 3x + 7$$

Change signs and multiply the divisor by 4 ;

$$4x^2+3x-7)\,4x^3+16x^2-20\,(x$$
$$4x^3+3x^2-7x$$
$$\overline{\quad 13x^2+7x-20}$$

Multiply by 4, and continue the division ;

$$52x^2+28x-80\,(13$$
$$52x^2+39x-91$$
$$\overline{\quad -11x+11}$$

Change signs and divide by 11 ;

$$x-1)\,4x^2+3x-7\,(4x+7$$

Hence $x-1$ is the H. C. F. of the first two expressions ; and since $x-1$ divides x^3+4x^2-8x+3 exactly, it is the H. C. F. required.

3. $2x^2+x-1=(2x-1)(x+1)$; $x^2+5x+4=(x+4)(x+1)$;
$x^3+1=(x+1)(x^2-x+1)$.

4. $y^3-y^2-y+1=(y^3-y^2)-(y-1)=y^2(y-1)-(y-1)=(y^2-1)(y-1)$
$3y^2-2y-1=(3y+1)(y-1)$
$y^3-y^2+y-1=y^2(y-1)+(y-1)=(y^2+1)(y-1)$.

5. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$x^3-4x^2+9x-10$$
$$\overline{\quad 6x^2-12x+30}$$

Divide by 6 ; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

$$x^3-2x^2+5x$$
$$\overline{\quad -2x^2+4x-10}$$
$$-2x^2+4x-10$$
$$\overline{\qquad\qquad}$$

Hence x^2-2x+5 is the H. C. F. of the first two expressions, and as it also divides $x^3+5x^2-9x+35$ exactly, it is the H. C. F. required.

6. $x^3-7x^2+16x-12)\,3x^3-14x^2+16x\,(3$

$$\begin{array}{r} 3x^3-21x^2+48x-36 \\ \hline 7x^2-32x+36 \end{array}$$

Multiply the divisor by 7;

$$7x^2-32x+36)\,7x^3-49x^2+112x-84\,(x$$

$$\begin{array}{r} 7x^3-32x^2+36x \\ \hline -17x^2+76x-84 \end{array}$$

Multiply by -7, and continue the division;

$$\begin{array}{r} 119x^2-532x+588\,(17 \\ 119x^2-544x+612 \\ \hline 12x-24 \end{array}$$

Divide by 12; $x-2)\,7x^2-32x+36\,(7x-18$

Hence $x-2$, which also divides $5x^3-10x^2+7x-14$, is the H. C. F.

7. $y^3-5y^2+11y-15)\,y^3-y^2+3y+5\,(1$

$$\begin{array}{r} y^3-5y^2+11y-15 \\ \hline 4y^2-8y+20 \end{array}$$

Divide by 4; $y^2-2y+5)\,y^3-5y^2+11y-15\,(y-3$

$$\begin{array}{r} y^3-2y^2+5y \\ \hline -3y^2+6y-15 \\ -3y^2+6y-15 \\ \hline \end{array}$$

Hence as y^2-2y+5 divides $2y^3-7y^2+16y-15$ exactly, it is the H. C. F.

XXXIX.

11. $\dfrac{a^2}{a^2+ab}=\dfrac{a^2}{a(a+b)}=\dfrac{a}{a+b}$.

12. $\dfrac{14m^2x}{21m^3p-7mx}=\dfrac{14m^2x}{7m(3m^2p-x)}=\dfrac{2mx}{3m^2p-x}$.

13. $\dfrac{xy}{3xy^2-5x^2yz}=\dfrac{xy}{xy(3y-5xz)}=\dfrac{1}{3y-5xz}$.

14. $\dfrac{4ax+2x^2}{8ax^3-2x^2}=\dfrac{2x(2a+x)}{2x(4ax^2-x)}=\dfrac{2a+x}{4ax^2-x}.$

15. $\dfrac{ay+y^2}{abc+bcy}=\dfrac{y(a+y)}{bc(a+y)}=\dfrac{y}{bc}.$

16. $\dfrac{4a^2x+6a^2y}{8x^2-18y^2}=\dfrac{2a^2(2x+3y)}{2(2x+3y)(2x-3y)}=\dfrac{a^2}{2x-3y}.$

17. $\dfrac{12ab^2-6ab}{8b^2c-2c}=\dfrac{6ab(2b-1)}{2c(2b+1)(2b-1)}=\dfrac{3ab}{2bc+c}.$

18. $\dfrac{c^2-4a^2}{c^2+4ac+4a^2}=\dfrac{(c+2a)(c-2a)}{(c+2a)(c+2a)}=\dfrac{c-2a}{c+2a}.$

19. $\dfrac{3x^4+3x^2y^2}{5x^4+5x^2y^2}=\dfrac{3x^2(x^2+y^2)}{5x^2(x^2+y^2)}=\dfrac{3}{5}.$

20. $\dfrac{10x-10y}{4x^2-8xy+4y^2}=\dfrac{10(x-y)}{4(x-y)(x-y)}=\dfrac{5}{2x-2y}.$

21. $\dfrac{ax+by}{7a^2x^2-7b^2y^2}=\dfrac{ax+by}{7(ax+by)(ax-by)}=\dfrac{1}{7ax-7by}.$

22. $\dfrac{6ab+8cd}{27a^2b^2x-48c^2d^2x}=\dfrac{2(3ab+4cd)}{3x(3ab+4cd)(3ab-4cd)}=\dfrac{2}{9abx-12cdx}.$

23. $\dfrac{xy-xyz}{2az-2az^2}=\dfrac{xy(1-z)}{2az(1-z)}=\dfrac{xy}{2az}.$

24. $\dfrac{7ab^3x^3-7ab^3y^3}{14a^3bcx^3-14a^3bcy^3}=\dfrac{7ab^3(x^3-y^3)}{14a^3bc(x^3-y^3)}=\dfrac{b^2}{2a^2c}.$

25. $\dfrac{5x^9+45dx^2}{10cx^9+90cdx^2}=\dfrac{5x^2(x^7+9d)}{10cx^2(x^7+9d)}=\dfrac{1}{2c}.$

26. $\dfrac{10a^2+20ab+10b^2}{5a^3+5a^2b}=\dfrac{10(a+b)(a+b)}{5a^2(a+b)}=\dfrac{2a+2b}{a^2}.$

27. $\dfrac{4x^2-8xy+4y^2}{48(x-y)^2}=\dfrac{4(x-y)^2}{48(x-y)^2}=\dfrac{1}{12}.$

28. $\dfrac{3mx+5nx^2}{3my+5nxy}=\dfrac{x(3m+5nx)}{y(3m+5nx)}=\dfrac{x}{y}.$

XL.

1. $\dfrac{a^2+7a+10}{a^2+5a+6}=\dfrac{(a+5)(a+2)}{(a+3)(a+2)}=\dfrac{a+5}{a+3}.$

2. $\dfrac{x^2-9x+20}{x^2-7x+12}=\dfrac{(x-4)(x-5)}{(x-4)(x-3)}=\dfrac{x-5}{x-3}.$

3. $\dfrac{x^2-2x-3}{x^2-10x+21}=\dfrac{(x-3)(x+1)}{(x-3)(x-7)}=\dfrac{x+1}{x-7}.$

4. $\dfrac{x^2-18xy+45y^2}{x^2-8xy-105y^2}=\dfrac{(x-3y)(x-15y)}{(x+7y)(x-15y)}=\dfrac{x-3y}{x+7y}.$

5. $\dfrac{x^4+x^2+1}{x^2+x+1}=\dfrac{(x^2+x+1)(x^2-x+1)}{x^2+x+1}=x^2-x+1.$

6. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}=\dfrac{(x^3+y^3)(x^3+y^3)}{(x^3+y^3)(x^3-y^3)}=\dfrac{x^3+y^3}{x^3-y^3}.$

7. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$\begin{array}{r} x^3-4x^2+9x-10 \\ \hline 6x^2-12x+30 \end{array}$$

Divide by 6; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

Hence x^2-2x+5 is the H. C. F.

8. $x^3-5x^2+11x-15)\,x^3-x^2+3x+5\,(1$

$$\begin{array}{r} x^3-5x^2+11x-15 \\ \hline 4x^2-8x+20 \end{array}$$

Divide by 4; $x^2-2x+5)\,x^3-5x^2+11x-15\,(x-3$

Hence x^2-2x+5 is the H. C. F.

9. $x^3-8x^2+21x-18)\,3x^3-16x^2+21x\,(3$

$$\begin{array}{r} 3x^3-24x^2+63x-54 \\ \hline 8x^2-42x+54 \end{array}$$

Divide by 2; $4x^2-21x+27$

Multiply divisor by 4 ; $4x^2-21x+27)4x^3-32x^2+84x-72(x$

$$4x^3-21x^2+27x$$

$$\overline{-11x^2+57x-72}$$

Multiply by -4, and continue the division ;

$$4x^2-21x+27)44x^2-228x+288(11$$

$$44x^2-231x+297$$

$$\overline{3x-9}$$

Divide by 3 ; $x-3)4x^2-21x+27(4x-9$

Hence $x-3$ is the H. C. F.

10. The H. C. F. is $x-2$, see the work in xxxviii. 6.

11. $x^4-x^3y-xy^3-y^4)x^4+x^3y+xy^3-y^4(1$

$$x^4-x^3y-xy^3-y^4$$

$$\overline{2x^3y+2xy^3}$$

Divide by $2xy$; $x^2+y^2)x^4-x^3y-xy^3-y^4(x^2-xy-y^2.$

12. $a^3-3a+2)a^3+4a^2-5(1$

$$a^3-3a+2$$

$$\overline{4a^2+3a-7}$$

Multiply divisor by 4 ; $4a^2+3a-7)4a^3-12a+8(a$

$$4a^3+3a^2-7a$$

$$\overline{-3a^2-5a+8}$$

Multiply by -4 and continue the division ;

$$12a^2+20a-32(3$$

$$12a^2+9a-21$$

$$\overline{11a-11}$$

Divide by 11 ; $a-1)4a^2+3a-7(4a+7.$

13. $b^3-6b+5)b^3+4b^2-5b(1$

$$b^3-6b+5$$

$$\overline{4b^2+b-5}$$

Multiply divisor by 4; $4b^2+b-5)\,4b^3-24b+20\,(b$

$$4b^3+b^2-5b$$
$$\overline{-b^2-19b+20}$$

Change signs; $b^2+19b-20)\,4b^2+b-5\,(4$

$$4b^2+76b-80$$
$$\overline{-75b+75}$$

Hence the H. C. F. is found to be $b-1$.

14. $m^3-7m+6)\,m^3+3m^2-4m\,(1$

$$m^3-7m+6$$
$$\overline{3m^2+3m-6}$$

Divide by 3; $m^2+m-2)\,m^3-7m+6\,(m-1$

$$m^3+m^2-2m$$
$$\overline{-m^2-5m+6}$$
$$-m^2-m+2$$
$$\overline{-4m+4}$$

Hence the H, C. F. is found to be $m-1$.

15. $a^3+1)\,a^3+2a^2+2a+1\,(1$

$$a^3\qquad\qquad+1$$
$$\overline{2a^2+2a}$$

Divide by $2a$, and H. C. F. is found to be $a+1$.

16. $\dfrac{3ax^2-13ax+14a}{7x^3-17x^2+6x}=\dfrac{a(3x^2-13x+14)}{x(7x^2-17x+6)}$

$3x^2-13x+14)\,21x^2-51x+18\,(7$

$$21x^2-91x+98$$
$$\overline{40x-80}$$

Whence H. C. F. of $3x^2-13x+14$ and $7x^2-17x+6$ is found to be $x-2$.

17. $\dfrac{14x^2-34x+12}{9ax^2-39ax+42a}=\dfrac{2(7x^2-17x+6)}{3a(3x^2-13x+14)}$;

and as in the preceding question, the H. C. F. is found to be $x-2$.

18. $$\frac{10a-24a^2+14a^3}{15-24a+3a^2+6a^3}=\frac{2a(5-12a+7a^2)}{3(5-8a+a^2+2a^3)}$$

$$5-12a+7a^2)\,5-8a+a^2+2a^3\,(1$$
$$5-12a+7a^2$$
$$\overline{4a-6a^2+2a^3}$$

Divide by $2a$; $2-3a+a^2=(1-a)(1-2a)$

Hence the H. C. F. is found to be $1-a$.

19. $$\frac{2ab^3+ab^2-8ab+5a}{7b^3-12b^2+5b}=\frac{a(2b^3+b^2-8b+5)}{b(7b^2-12b+5)}$$

$$7b^2-12b+5)\,14b^3+7b^2-56b+35\,(2b$$
$$14b^3-24b^2+10b$$
$$\overline{31b^2-66b+35}$$
$$7$$
$$\overline{217b^2-462b+245\,(31}$$
$$217b^2-372b+155$$
$$\overline{-90b+90}$$

Hence the H. C. F. is found to be $b-1$.

20. $$a^3-4a^2+6a-4)\,a^3-3a^2+3a-2\,(1$$
$$a^3-4a^2+6a-4$$
$$\overline{a^2-3a+2)\,a^3-4a^2+6a-4\,(a-1}$$
$$a^3-3a^2+2a$$
$$\overline{-a^2+4a-4}$$
$$-a^2+3a-2$$
$$\overline{a-2}$$

Hence we find $a-2$ to be the H. C. F.

21. $$\frac{3x^2+2x-1}{x^3+x^2-x-1}=\frac{(3x-1)(x+1)}{(x^2-1)(x+1)}=\frac{3x-1}{x^2-1}.$$

22. $$\frac{a^2-a-20}{a^2+a-12}=\frac{(a-5)(a+4)}{(a-3)(a+4)}=\frac{a-5}{a-3}.$$

23. $x^3 - 3x^2 + 4x - 2)\,x^3 - x^2 - 2x + 2\,(1$

$$\begin{array}{r} x^3 - 3x^2 + 4x - 2 \\ \hline 2x^2 - 6x + 4 \end{array}$$

Divide by 2 ; $x^2 - 3x + 2)\,x^3 - 3x^2 + 4x - 2\,(x$

$$\begin{array}{r} x^3 - 3x^2 + 2x \\ \hline 2x - 2 \end{array}$$

Hence we find $x - 1$ to be the H. C. F.

24. $$\frac{x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + z^2 - 2yz + y^2 + x^2 - 2xz + z^2 + y^2 - 2xy + x^2}{x^2 + y^2 + z^2}$$

$$= \frac{3x^2 + 3y^2 + 3z^2}{x^2 + y^2 + z^2} = 3.$$

25. Multiply the numerator by 7 ;

$7x^3 - 19x^2 + 17x - 5)\,14x^4 - 7x^3 - 63x^2 + 91x - 35\,(2x$

$$\begin{array}{r} 14x^3 - 38x^3 + 34x^2 - 10x \\ \hline 31x^3 - 97x^2 + 101x - 35 \\ 7 \\ \hline 217x^3 - 679x^2 + 707x - 245\,(31 \\ 217x^3 - 589x^2 + 527x - 155 \\ \hline -90x^2 + 180x - 90 \end{array}$$

Divide by -90 ; $x^2 - 2x + 1)\,7x^3 - 19x^2 + 17x - 5\,(7x - 5$

$$\begin{array}{r} 7x^3 - 14x^2 + 7x \\ \hline -5x^2 + 10x - 5 \\ -5x^2 + 10x - 5 \\ \hline \end{array}$$

Hence $x^2 - 2x + 1$ is the H. C. F.

26. $8x^4 - 30x^3 + 31x^2 - 12)\,16x^4 - 53x^2 + 45x + 6\,(2$

$$\begin{array}{r} 16x^4 - 60x^3 + 62x^2 - 24 \\ \hline 60x^3 - 115x^2 + 45x + 30 \end{array}$$

Divide by 5, and multiply the divisor by 3;

$$12x^3-23x^2+9x+6)\,24x^4-90x^3+93x^2-36\,(2x$$
$$12x^4-46x^3+18x^2+12x$$
$$\overline{-44x^3+75x^2-12x-36}$$

Multiply by -3, and continue the division;

$$132x^3-225x^2+36x+108\,(11$$
$$132x^3-253x^2+99x+\ 66$$
$$\overline{28x^2-63x+\ 42}$$

Divide by 7; $4x^2-9x+6)\,12x^3-23x^2+9x+6\,(3x+1$

Hence $4x^2-9x+6$ is the H. C. F.

27. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}=\dfrac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}.$

28. $6x^3-23x^2+16x-3)\,6x^3-17x^2+11x-2\,(1$
$$6x^3-23x^2+16x-3$$
$$\overline{6x^2-\ 5x+1}$$

$6x^2-5x+1)\,6x^3-23x^2+16x-3\,(x-3$

Hence $6x^2-5x+1$ is the H. C. F.

29. $x^3-6x^2+11x-6)\,x^3-2x^2-\ x\ +2\,(1$
$$x^3-6x^2+11x-6$$
$$\overline{4x^2-12x+8}$$

Divide by 4; $x^2-3x+2)x^3-6x^2+11x-6\,(x-3$

Hence x^2-3x+2 is the H. C. F.

30. $m^3+m^2+m-3)\,m^3+3m^2+5m+3\,(1$
$$m^3+\ m^2+\ m-3$$
$$\overline{2m^2+4m+6}$$

Divide by 2; $m^2+2m+3)\,m^3+m^2+m-3\,(m-1$

Hence m^2+2m+3 is the H. C. F.

31. $\dfrac{x^5+5x^4-x^2-5x}{x^4+3x^3-x-3}=\dfrac{x^4(x+5)-x(x+5)}{x^3(x+3)-(x+3)}=\dfrac{(x+5)(x^3-1)x}{(x+3)(x^3-1)}=\dfrac{x^2+5x}{x+3}.$

32. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}=\dfrac{a^2-(b^2+2bc+c^2)}{(a^2+2ab+b^2)-c^2}=\dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}$

$=\dfrac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}=\dfrac{a-b-c}{a+b-c}.$

33. Multiply the numerator by 3;

$9a^2+3ab-2b^2)45a^2+3ab-6b^2(5$
$45a^2+15ab-10b^2$
$-12ab+4b^2$

Divide by $-4b$; $3a-b)9a^2+3ab-2b^2(3a+2b$

Hence $3a-b$ is the H. C. F.

34. $x^2-7x+10=(x-5)(x-2)$

$x-5$ is clearly not a factor of the denominator;

$x-2)2x^2-x-6(2x+3$

Hence $x-2$ is the H. C. F.

35. $x^3+3x^2+4x+12)x^3+4x^2+4x+3(1$
$x^3+3x^2+4x+12$
x^2-9

Divide by $x-3$; $x+3)x^3+3x^2+4x+12(x^2+4$

Hence $x+3$ is the H. C. F.

36. $2x^3-x-1)2x^4-2x^2-4x+4)x$
$2x^4-x^2-x$
$-x^2-3x+4$

Change signs; $x^2+3x-4=(x+4)(x-1)$

Divide by $x+4$; $x-1)2x^3-x-1(2x^2+2x+1$

Hence $x-1$ is the H. C. F.

37. $3x^2-4x-15)3x^3-6x^2-45x+108(x$
$3x^3-4x^2-15x$
$-2x^2-30x+108$

Divide by -2; $x^2+15x-54=(x+18)(x-3)$

Divide by $x+18$; $x-3)3x^2-4x-15(3x+5$

Hence $x-3$ is the H. C. F.

Divide by 5, and multiply the divisor by 3;

$$12x^3-23x^2+9x+6)\,24x^4-90x^3+93x^2-36\,(2x$$
$$12x^4-46x^3+18x^2+12x$$
$$\overline{-44x^3+75x^2-12x-36}$$

Multiply by -3, and continue the division;

$$132x^3-225x^2+36x+108\,(11$$
$$132x^3-253x^2+99x+66$$
$$\overline{28x^2-63x+42}$$

Divide by 7; $4x^2-9x+6)\,12x^3-23x^2+9x+6\,(3x+1$

Hence $4x^2-9x+6$ is the H. C. F.

27. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}=\dfrac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}.$

28. $6x^3-23x^2+16x-3)\,6x^3-17x^2+11x-2\,(1$
$$6x^3-23x^2+16x-3$$
$$\overline{6x^2-5x+1}$$

$6x^2-5x+1)\,6x^3-23x^2+16x-3\,(x-3$

Hence $6x^2-5x+1$ is the H. C. F.

29. $x^3-6x^2+11x-6)\,x^3-2x^2-x+2\,(1$
$$x^3-6x^2+11x-6$$
$$\overline{4x^2-12x+8}$$

Divide by 4; $x^2-3x+2)\,x^3-6x^2+11x-6\,(x-3$

Hence x^2-3x+2 is the H. C. F.

30. $m^3+m^2+m-3)\,m^3+3m^2+5m+3\,(1$
$$m^3+m^2+m-3$$
$$\overline{2m^2+4m+6}$$

Divide by 2; $m^2+2m+3)\,m^3+m^2+m-3\,(m-1$

Hence m^2+2m+3 is the H. C. F.

31. $\dfrac{x^5+5x^4-x^2-5x}{x^4+3x^3-x-3}=\dfrac{x^4(x+5)-x(x+5)}{x^3(x+3)-(x+3)}=\dfrac{(x+5)(x^3-1)x}{(x+3)(x^3-1)}=\dfrac{x^2+5x}{x+3}.$

32. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}=\dfrac{a^2-(b^2+2bc+c^2)}{(a^2+2ab+b^2)-c^2}=\dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}$

$=\dfrac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}=\dfrac{a-b-c}{a+b-c}.$

33. Multiply the numerator by 3;

$9a^2+3ab-2b^2)\,45a^2+3ab-6b^2\,(5$
$\underline{45a^2+15ab-10b^2}$
$-12ab+4b^2$

Divide by $-4b$; $3a-b)\,9a^2+3ab-2b^2\,(3a+2b$

Hence $3a-b$ is the H. C. F.

34. $x^2-7x+10=(x-5)(x-2)$

$x-5$ is clearly not a factor of the denominator;

$x-2)\,2x^2-x-6\,(2x+3$

Hence $x-2$ is the H. C. F.

35. $x^3+3x^2+4x+12)\,x^3+4x^2+4x+3\,(1$
$\underline{x^3+3x^2+4x+12}$
x^2-9

Divide by $x-3$; $x+3)\,x^3+3x^2+4x+12\,(x^2+4$

Hence $x+3$ is the H. C. F.

36. $2x^3-x-1)\,2x^4-2x^2-4x+4\,)x$
$\underline{2x^4-x^2-x}$
$-x^2-3x+4$

Change signs; $x^2+3x-4=(x+4)(x-1)$

Divide by $x+4$; $x-1)\,2x^3-x-1\,(2x^2+2x+1$

Hence $x-1$ is the H. C. F.

37. $3x^2-4x-15)\,3x^3-6x^2-45x+108\,(x$
$\underline{3x^3-4x^2-15x}$
$-2x^2-30x+108$

Divide by -2; $x^2+15x-54=(x+18)(x-3)$

Divide by $x+18$; $x-3)\,3x^2-4x-15\,(3x+5$

Hence $x-3$ is the H. C. F.

38. $3x^3 + x^2 - 5x + 21)\,6x^3 + 29x^2 + 26x - 21\,(2$

$$6x^3 + 2x^2 - 10x + 42$$

$$27x^2 + 36x - 63$$

Divide by 9; $3x^2 + 4x - 7)\,3x^3 + x^2 - 5x + 21\,(x - 1$

$$3x^3 + 4x^2 - 7x$$

$$-3x^2 + 2x + 21$$

$$-3x^2 - 4x + 7$$

$$6x + 14$$

Divide by 2; $3x + 7)\,3x^2 + 4x - 7\,(x - 1$

Hence $3x + 7$ is the H. C. F.

39. $4x^3 - 3x^2 - 8x - 1)\,4x^4 - 4x^3 - 16x^2 - 4x + 4\,(x$

$$4x^4 - 3x^3 - 8x^2 - x$$

$$-x^3 - 8x^2 - 3x + 4$$

Change signs; $x^3 + 8x^2 + 3x - 4)\,4x^3 - 3x^2 - 8x - 1\,(4$

$$4x^3 + 32x^2 + 12x - 16$$

$$-35x^2 - 20x + 15$$

Divide by -5; $7x^2 + 4x - 3 = (7x - 3)(x + 1)$

Divide by $7x - 3$; $x + 1)\,x^3 + 8x^2 + 3x - 4\,(x^2 + 7x - 4$

Hence $x + 1$ is the H. C. F.

40. $a^3 - 7a^2 + 16a - 12)\,3a^3 - 14a^2 + 16a\,(3$

$$3a^3 - 21a^2 + 48a - 36$$

$$7a^2 - 32a + 36$$

$7a^2 - 32a + 36)\,7a^3 - 49a^2 + 112a - 84\,(a$

$$7a^3 - 32a^2 + 36a$$

$$-17a^2 + 76a - 84$$

$$7$$

$$-119a^2 + 532a - 588\,(-17$$

$$-119a^2 + 544a - 612$$

$$-12a + 24$$

Hence we find $a - 2$ to be the H. C. F.

XLII.

1. $\dfrac{a-b}{a^2+ab}\times\dfrac{a^2-b^2}{a^2-ab}=\dfrac{(a-b)(a+b)(a-b)}{a(a+b)a(a-b)}=$ etc.

2. $\dfrac{x^2+4x}{x^2-3x}\times\dfrac{4x^2-12x}{3x^2+12x}=\dfrac{x(x+4)4x(x-3)}{x(x-3)3x(x+4)}=$ etc.

3. $\dfrac{x^2+3x+2}{x^2-5x+6}\times\dfrac{x^2-7x+12}{x^2+x}=\dfrac{(x+2)(x+1)(x-3)(x-4)}{(x-2)(x-3)x(x+1)}=$ etc.

4. $\dfrac{x^2+x-2}{x^2-7x}\times\dfrac{x^2-13x+42}{x^2+2x}=\dfrac{(x+2)(x-1)(x-7)(x-6)}{x(x-7)x(x+2)}=$ etc.

5. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}=\dfrac{(x-6)(x-5)x(x-3)}{(x-3)(x-3)x(x-5)}=$ etc.

6. $\dfrac{x^2-4}{x^2+5x}\times\dfrac{x^2-25}{x^2+2x}=\dfrac{(x+2)(x-2)(x+5)(x-5)}{x(x+5)x(x+2)}=$ etc.

7. $\dfrac{(a-3)(a-1)}{(a-4)(a-1)}\times\dfrac{(a-5)(a-4)}{(a-7)(a-3)}\times\dfrac{a(a-7)}{a(a-5)}=1.$

8. $\dfrac{(b-6)(b-1)}{(b+4)(b-1)}\times\dfrac{(b+4)(b+6)}{(b-8)(b-6)}\times\dfrac{b^2(b-8)}{b(b+6)}=b.$

9. $\dfrac{(x+y)(x-y)}{(x-y)(x-2y)}\times\dfrac{y(x-2y)}{x(x+y)}\times\dfrac{x(x-y)}{(x-y)(x-y)}=\dfrac{y}{x-y}.$

10. $\dfrac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)}\times\dfrac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)}=\dfrac{c-a+b}{c-a-b}.$

11. $\dfrac{(x-m+n)(x-m-n)}{(x-n+m)(x-n-m)}\times\dfrac{(x+n-m)(x-n+m)}{(x+m-n)(x-m+n)}=\dfrac{x-m+n}{x+m-n}.$

12. $\dfrac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)}\times\dfrac{(a-b+d-c)(a-b-d+c)}{(a-c+d-b)(a-c-d+b)}=1.$

13. $\dfrac{(x^2-2xy+y^2)-z^2}{(x^2+2xy+y^2)-z^2}\times\dfrac{x+y-z}{x-y+z}=\dfrac{(x-y+z)(x-y-z)(x+y-z)}{(x+y+z)(x+y-z)(x-y+z)}=$ etc.

XLIII.

1. $\dfrac{2a}{x}\times\dfrac{5c}{3b}=\dfrac{10ac}{3bx}.$

2. $\dfrac{15y}{14z}\times\dfrac{7z}{5y^2}=\dfrac{3}{2y}.$

3. $\dfrac{8x^4y}{15ab^3}\times\dfrac{30ab^2}{2x^3}=\dfrac{8xy}{b}.$

4. $\dfrac{4a}{nx}\times\dfrac{1}{3ab}=\dfrac{4}{3bnx}.$

5. $\frac{3p}{2(p-1)} \times \frac{p-1}{2p} = \frac{3}{4}$.

6. $1 \times \frac{5x}{4a} = \frac{5x}{4a}$.

7. $\frac{5x}{7} \times \frac{1}{2} = \frac{5x}{14}$.

8. $\frac{1}{(x-1)(x-2)} \times \frac{x-1}{1} = \frac{1}{x-2}$.

9. $\frac{1}{(x-15)(x-2)} \times \frac{x-15}{1} = \frac{1}{x-2}$.

XLV.

1. $x^2 = x \times x$; $ax + x^2 = x(a+x)$; $\therefore$ L. C. M. is $x \times x(a+x)$.

2. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $\therefore$ L. C. M. is $(x+1)(x-1)x$.

3. $a^2 - b^2 = (a+b)(a-b)$; $a^2 + ab = a(a+b)$; $\therefore$ L. C. M. is $(a+b)(a-b)a$.

4. $4x^2 - 1 = (2x-1)(2x+1)$; $\therefore$ L. C. M. is $4x^2 - 1$.

5. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$; $\therefore$ L. C. M. is $a^3 + b^3$.

6. $x^2 - 1 = (x+1)(x-1)$; $\therefore$ L. C. M. is $x^2 - 1$.

7. $x^3 - 1 = (x-1)(x^2+x+1)$; $\therefore$ L. C. M. is $(x+1)(x^3-1)$.

8. $x^3 + 1 = (x+1)(x^2-x+1)$; $\therefore$ L. C. M. is $(x^2+1)(x^3+1)$.

9. $x^2 - 1 = (x+1)(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)(x^2+x+1)$.

10. $x^4 - 1 = (x^2+1)(x^2-1)$; $\therefore$ L. C. M. is $x^4 - 1$.

11. $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$; $x^3 + 1 = (x+1)(x^2-x+1)$;
$\therefore$ L. C. M. is $x(x-1)(x^2+x+1)(x+1)(x^2-x+1)$, etc.

12. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)x(x^2+x+1)$, etc.

13. $4a^2 - 1 = (2a+1)(2a-1)$; $8a^3 + 1 = (2a+1)(4a^2-2a+1)$;
$\therefore$ L. C. M. is $(2a+1)(2a-1)(4a^2-2a+1)$, etc.

14. $2x^2+2xy=2x(x+y)$; $\therefore$ L. C. M. is $2x^2+2xy$.

15. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is $(a+b)(a+b)(a-b)$.

16. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is a^2-b^2.

17. $2(1-x^2)=2(1+x)(1-x)$; $\therefore$ L. C. M. is $4(1+x)(1-x)$.

18. $x^3-1=(x-1)(x^2+x+1)$; $\therefore$ L. C. M. is x^3-1.

19. L. C. M. is $(a-b)(a-c)(b-c)$.

20. L. C. M. is $(x+1)(x+2)(x+3)$.

21. $x^2-y^2=(x+y)(x-y)$;
$\therefore$ L. C. M. is $(x+y)(x-y)(x+y)(x-y)$.

22. $a^2-1=(a+1)(a-1)$; $\therefore$ L. C. M. is $(a+3)(a+1)(a-1)$.

23. $x(x^2-y^2)=x(x+y)(x-y)$; $\therefore$ L. C. M. is $x^2(x-y)(x+y)$.

24. L. C. M. is $(x+1)(x+3)(x+2)(x+4)$.

25. $x^2-y^2=(x+y)(x-y)$; $12(x^3+y^3)=12(x+y)(x^2-xy+y^2)$;
$\therefore$ L. C. M. is $12(x+y)(x-y)(x-y)(x^2-xy+y^2)$.

26. $6(x^2+xy)=6x(x+y)$; $8(xy-y^2)=8y(x-y)$;
$10(x^2-y^2)=10(x+y)(x-y)$;
$\therefore$ L. C. M. is $120xy(x+y)(x-y)$.

XLVI.

1. $x^2+5x+6=(x+2)(x+3)$
$x^2+6x+8=(x+2)(x+4)$.

2. $a^2-a-20=(a-5)(a+4)$
$a^2+a-12=(a-3)(a+4)$.

3. $x^2+3x+2=(x+1)(x+2)$
$x^2+4x+3=(x+1)(x+3)$.

4. $x^2+11x+30=(x+5)(x+6)$
$x^2+12x+35=(x+5)(x+7)$.

5. $x^2-\ \ 9x-22=(x-11)(x+2)$
$x^2-13x+22=(x-11)(x-2)$.

6. $2x^2+3x+1=(2x+1)(x+1)$
$x^2-x-2=(x-2)(x+1)$.

7. $x^3+x^2y+xy+y^2=(x^2+y)(x+y)$
$x^4-y^4=(x^2+y^2)(x+y)(x-y)$.

8. $x^2-8x+15=(x-3)(x-5)$
$x^2+2x-15=(x-3)(x+5)$.

9. $21x^2-26x+8)\,21x^3-12x^2-63x+36\,(x$
$21x^3-26x^2+8x$
$\overline{\qquad 14x^2-71x+36}$
3
$\overline{42x^2-213x+108}\,(2$
$42x^2-52x+16$
$\overline{\qquad -161x+92}$

Divide by -23; $7x-4)\,21x^2-26x+8\,(3x-2$
$\therefore$ L. C. M. is $(3x-2)(7x^3-4x^2-21x+12)$
or $(3x-2)(7x-4)(x^2-3)$.

10. $x^3+x^2y+xy^2+y^3=(x^2+y^2)(x+y)$
$x^3-x^2y+xy^2-y^3=(x^2+y^2)(x-y)$.

11. $a^3-2a^2b-ab^2+2b^3)\,a^3+2a^2b-ab^2-2b^3\,(1$
$a^3-2a^2b-ab^2+2b^3$
$\overline{\qquad 4a^2b \qquad -4b^3}$

Divide by $4b$; $a^2-b^2)\,a^3-2a^2b-ab^2+2b^3\,(a-2b$
$\therefore$ L. C. M. is $(a-2b)(a^3+2a^2b-ab^2-2b^3)$
or $(a-2b)(a^2-b^2)(a+2b)$.

XLVII.

1. $x^2-3x+2=(x-1)(x-2)$
$x^2-4x+3=(x-1)(x-3)$
$x^2-5x+4=(x-1)(x-4).$

2. $x^2+5x+4=(x+1)(x+4)$
$x^2+4x+3=(x+1)(x+3)$
$x^2+7x+12=(x+3)(x+4).$

3. $x^2-\ \ 9x+20=(x-4)(x-5)$
$x^2-12x+35=(x-7)(x-5)$
$x^2-11x+28=(x-4)(x-7)$

4. $6x^2-x-2)\,42x^2-34x+\ \ 4\,(7$
$42x^2-\ \ 7x-14$
$\overline{\qquad -27x+18}$

Divide by -9; $3x-2)\,6x^2-x-2\,(2x+1.$
Hence L. C. M. of first two expressions is
$(2x+1)(21x^2-17x+2)$, or $42x^3-13x^2-13x+2.$
We have now to find the L. C. M. of this and $14x^2+5x-1$.
$14x^2+5x-1)\,42x^3-13x^2-13x+2\,(3x-2$
$42x^3+15x^2-\ \ 3x$
$\overline{\qquad -28x^2-10x+2}$
$-28x^2-10x+2$
$\therefore$ L. C. M. is $(3x-2)(14x^2+5x-1)$, etc.

5. $x^2-1=(x+1)(x-1)$
$x^2+2x-3=(x+3)(x-1)$
$6x^2-x-2=(3x-2)(2x+1).$

6. $x^3-27=(x-3)(x^2+3x+9)$
$x^2-15x+36=(x-3)(x-12)$
$x^3-3x^2-2x+6=x^2(x-3)-2(x-3)=(x^2-2)(x-3).$

XLVIII.

1. L. C. D. 20; new numerators $15x$, $16x$.

2. L. C. D. 18; new numerators $9x-21$, $4x-9$.

3. L. C. D. $10x^2$; new numerators $4x-8y$, $3x^2-8xy$.

4. L. C. D. $10a^2$; new numerators $20a+25b$, $6a^2-8ab$.

5. L. C. D. $60a^2c$; new numerators $48a^2-60ac$, $15a-10c$.

6. L. C. D. a^3b^2; new numerators $ab-b^2$, a^4-a^3b.

7. L. C. D. $1-x^2$; new numerators $3-3x$, $3+3x$.

8. L. C. D. $1-y^4$; new numerators $2+2y^2$, $2-2y^2$.

9. L. C. D. $1-x^2$; new numerators $5+5x$, 6.

10. L. C. D. $c(b+x)$; new numerators $ab+ax$, b.

11. L. C. D. $(a-b)(b-c)(a-c)$; new numerators $a-c$, $b-c$.

12. L. C. D. $abc(a-b)(a-c)(b-c)$; new numerators $bc-c^2$, $ab-b^2$.

XLIX.

1. $\dfrac{12x+21}{15}+\dfrac{3x-4}{15}=\dfrac{15x+17}{15}$.

2. $\dfrac{36a-48b}{84}-\dfrac{56a-28b+28c}{84}+\dfrac{91a-28c}{84}=\dfrac{71a-20b-56c}{84}$.

3. $\dfrac{24x-18y}{42}+\dfrac{9x+21y}{42}-\dfrac{10x-4y}{42}+\dfrac{9x+2y}{42}=\dfrac{32x+9y}{42}$.

4. $\dfrac{30x-20y}{50x}+\dfrac{25x-35y}{50x}+\dfrac{16x^2+4xy}{50x}=\dfrac{16x^2+55x+4xy-55y}{50x}$.

5. $\dfrac{16x^3-28y^2}{12x^3}+\dfrac{6x^3-16xy}{12x^3}+\dfrac{5x^3-2x^2y}{12x^3}=\dfrac{27x^3-2x^2y-16xy-28y^2}{12x^3}$.

6. $\dfrac{180a^2+225b^2}{90b^2}+\dfrac{54ab+36b^2}{90b^2}+\dfrac{70b^2-20ab^2}{90b^2}=\dfrac{180a^2+54ab+331b^2-20ab^2}{90b^2}.$

7. $\dfrac{80x^3+100x^2}{60x^2}-\dfrac{36x^2-84x}{60x^2}+\dfrac{45}{60x^2}=\dfrac{80x^3+64x^2+84x+45}{60x^2}.$

8. $\dfrac{70a^2+28ab}{42ac}-\dfrac{84c^2-63bc}{42ac}+\dfrac{18ab-21bc}{42ac}=\dfrac{70a^2+46ab+42bc-84c^2}{42ac}$, etc.

9. $\dfrac{2ac+5c^2}{a^2c^2}+\dfrac{4a^2c-3ac^2}{a^2c^2}-\dfrac{5ac-2c^2}{a^2c^2}=\dfrac{4a^2c-3ac^2-3ac+7c^2}{a^2c^2}$, etc.

10. $\dfrac{3x^2y^2-4xy}{x^3y^3}-\dfrac{5x^2y^2+7x^2}{x^3y^3}-\dfrac{6x^2y^2-11y^2}{x^3y^3}=\dfrac{11y^2-8x^2y^2-4xy-7x^2}{x^3y^3}.$

11. $\dfrac{abc^2-b^2c^2}{a^3b^2c^2}+\dfrac{4a^2bc-5ab^2c}{a^3b^2c^2}+\dfrac{3a^4-7a^3b}{a^3b^2c^2}=\dfrac{3a^4-7a^3b+4a^2bc-5ab^2c+abc^2-b^2c^2}{a^3b^2c^2}.$

L.

1. $\dfrac{x+5+x-6}{(x-6)(x+5)}=\dfrac{2x-1}{(x-6)(x+5)}.$

2. $\dfrac{x-3-(x-7)}{(x-7)(x-3)}=\dfrac{4}{(x-7)(x-3)}.$

3. $\dfrac{1-x+1+x}{(1+x)(1-x)}=\dfrac{2}{(1+x)(1-x)}.$

4. $\dfrac{x^2+2xy+y^2-(x^2-2xy+y^2)}{(x+y)(x-y)}=\dfrac{4xy}{(x+y)(x-y)}.$

5. $\dfrac{1+x-2}{(1+x)(1-x)}=\dfrac{x-1}{(1+x)(1-x)}=\dfrac{-(1-x)}{(1+x)(1-x)}=\dfrac{-1}{1+x}$

6. $\dfrac{a(c+dx)-(ad-bc)x}{c(c+dx)}=\dfrac{ac+bcx}{c(c+dx)}=\dfrac{a+bx}{c+dx}.$

7. $\dfrac{x^2-xy+x^2+xy}{(x+y)(x-y)}=\dfrac{2x^2}{(x+y)(x-y)}.$

8. $\dfrac{x-y+x}{(x-y)^2}=\dfrac{2x-y}{(x-y)^2}.$

9. $\dfrac{2(x+a)+3a}{(x+a)^2}=\dfrac{2x+5a}{(x+a)^2}$

10. $\dfrac{a-x+a+x}{2a(a+x)(a-x)}=\dfrac{2a}{2a(a+x)(a-x)}=\dfrac{1}{(a+x)(a-x)}.$

LI.

1. $\dfrac{1-a+1+a}{1-a^2}+\dfrac{2a}{1-a^2}=\dfrac{2+2a}{1-a^2}=\dfrac{2(1+a)}{1-a^2}=\dfrac{2}{1-a}.$

2. $\dfrac{1+x-(1-x)}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x+2x^3+2x-2x^3}{1-x^4}=\dfrac{4x}{1-x^4}.$

3. $\dfrac{x+x^2-x^2}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x+x^3+x-x^3}{1-x^4}=\dfrac{2x}{1-x^4}.$

4. $\dfrac{a+b-(a-b)}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2b}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2a^2b+2b^3-(2a^2b-2b^3)}{a^4-b^4}$

$\dfrac{4b^3}{a^4-b^4}-\dfrac{4b^3}{a^4+b^4}=\dfrac{4a^4b^3+4b^7-(4a^4b^3-4b^7)}{a^8-b^8}=\dfrac{8b^7}{a^8-b^8}.$

5. $\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x^2\cancel{y}}{\cancel{y}(x+y)}=\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x}{x+y}=\dfrac{x^2+xy+y^2+xy}{y(x+y)}$

$=\dfrac{(x+y)^2}{y(x+y)}=\dfrac{x+y}{y}.$

6. $\dfrac{x^2-9+x^2-16}{(x+4)(x-3)}+\dfrac{x+5}{x+7}=\dfrac{2x^2-25}{(x+4)(x-3)}+\dfrac{x+5}{x+7}$

$=\dfrac{2x^3-25x+14x^2-175+(x+5)(x^2+x-12)}{(x+4)(x-3)(x+7)}$

$=\dfrac{2x^3-25x+14x^2-175+x^3+6x^2-7x-60}{(x+4)(x-3)(x+7)}$, etc.

7. $\dfrac{x^2-4x+3+x^2-4x+4}{(x-2)(x-3)}+\dfrac{x-3}{x-4}=\dfrac{2x^2-8x+7}{(x-2)(x-3)}+\dfrac{x-3}{x-4}$

$$=\frac{2x^3-16x^2+39x-28+(x-3)(x^2-5x+6)}{(x-2)(x-3)(x-4)}$$

$$=\frac{2x^3-16x^2+39x-28+x^3-8x^2+21x-18}{(x-2)(x-3)(x-4)}, \text{ etc.}$$

8. $\dfrac{3x-3a+4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}=\dfrac{(3x+a)(x-a)-5a^2}{(x-a)^3}=\dfrac{3x^2-2ax-a^2-5a^2}{(x-a)^3}$

$$=\frac{3x^2-2ax-6a^2}{(x-a)^3}.$$

9. $\dfrac{x+2-(x-1)}{(x-1)(x+2)}-\dfrac{3}{(x+1)(x+2)}=\dfrac{3}{(x-1)(x+2)}-\dfrac{3}{(x+1)(x+2)}$

$$=\frac{3x+3-(3x-3)}{(x-1)(x+1)(x+2)}=\frac{6}{(x-1)(x+1)(x+2)}.$$

10. $\dfrac{x+3}{(x+1)(x+2)(x+3)}-\dfrac{3}{(x+1)(x+2)(x+3)}=\dfrac{x}{(x+1)(x+2)(x+3)}.$

11. $\dfrac{x^2}{x^2-1}+\dfrac{x^2+x}{x^2-1}+\dfrac{x^2-x}{x^2-1}=\dfrac{3x^2}{x^2-1}.$

12. $\dfrac{a+e}{(a+c)(a+d)(a+e)}-\dfrac{a+d}{(a+c)(a+d)(a+e)}=\dfrac{e-d}{(a+c)(a+d)(a+e)}.$

13. $\dfrac{(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)}{(b+c)(c+a)(a+b)}$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b+c)(c+a)(a+b)}=0.$$

14. $\dfrac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}=\dfrac{x^2-2ax+a^2+x^2-2bx+b^2-a^2+2ab-b^2}{(x-a)(x-b)}$

$$=\frac{2(x^2-ax-bx+ab)}{x^2-ax-bx+ab}=2.$$

15. $$\frac{(x+y)^2-2xy}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}=\frac{x^2+y^2}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}$$

$$=\frac{(x^2+y^2)(x-y)+x^2y-x^3}{y(x^2-y^2)}=\frac{x^3+xy^2-x^2y-y^3+x^2y-x^3}{y(x^2-y^2)}$$

$$=\frac{y^2(x-y)}{y(x^2-y^2)}=\frac{y}{x+y}.$$

16. $$\frac{(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)}{(b-c)(c-a)(a-b)}$$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b-c)(c-a)(a-b)}=0.$$

17. $$\frac{x(x-y)}{x^3-y^3}+\frac{2xy}{x^3-y^3}=\frac{x^2+xy}{x^3-y^3}.$$

18. $$\frac{2b-2c+2a-2b}{(a-b)(b-c)}+\frac{2}{c-a}=\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)};$$

$$\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}.$$

$$=\frac{(2ac-2a^2-2c^2+2ac+2ab-2b^2-2ac+2bc)+(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2)}{(a-b)(b-c)(c-a)}$$

$$=0.$$

19. $$\frac{(a+b)^2-2ab}{b(a+b)}+\frac{a^2b-a^3}{b(a^2-b^2)}=\frac{(a^2+b^2)(a-b)+a^2b-a^3}{b(a^2-b^2)}$$

$$=\frac{a^3+ab^2-a^2b-b^3+a^2b-a^3}{b(a^2-b^2)}=\frac{b^2(a-b)}{b(a^2-b^2)}=\frac{b}{a+b}.$$

20. $$\frac{(n+3)-1-(n+2)}{(n+1)(n+2)(n+3)}=0.$$

21. $$\frac{(a^2-bc)(b+c)+(b^2-ac)(a+c)+(c^2-ab)(a+b)}{(a+b)(a+c)(b+c)}$$

$$=\frac{a^2b+a^2c-b^2c-bc^2+ab^2-a^2c+b^2c-ac^2+ac^2-a^2b+bc^2-ab^2}{(a+b)(a+c)(b+c)}$$

$$=0.$$

LII.

1. $$\frac{x}{x-y}+\frac{x-y}{y-x}=\frac{x}{x-y}+\frac{-x+y}{x-y}=\frac{y}{x-y}.$$

2. $$\frac{(3+2x)(2+x)}{4-x^2}-\frac{(2-3x)(2-x)}{4-x^2}+\frac{x^2-16x}{4-x^2}$$
$$=\frac{6+7x+2x^2-(4-8x+3x^2)+x^2-16x}{4-x^2}=\frac{2-x}{4-x^2}=\frac{1}{2+x}.$$

3. $$\frac{x}{x+1}-\frac{-x}{x-1}+\frac{x^2}{x^2-1}=\frac{x^2-x-(-x^2-x)+x^2}{x^2-1}=\frac{3x^2}{x^2-1}.$$

4. $$\frac{1}{6(y+1)}-\frac{1}{2(y-1)}+\frac{4}{3(1-y^2)}=\frac{y-1-3(y+1)}{6(y^2-1)}+\frac{-4}{3(y^2-1)}$$
$$=\frac{y-1-3y-3-8}{6(y^2-1)}=\frac{-2y-12}{6(y^2-1)}=\frac{2y+12}{6(1-y^2)}=\frac{y+6}{3(1-y^2)}.$$

5. $$\frac{1}{(m-2)(m-3)}+\frac{-2}{(m-1)(m-3)}+\frac{1}{(m-1)(m-2)}$$
$$=\frac{m-1-2(m-2)+m-3}{(m-1)(m-2)(m-3)}=0.$$

6. $$\frac{1}{(a-b)(x+b)}+\frac{-1}{(a-b)(x+a)}=\frac{x+a-(x+b)}{(a-b)(x+a)(x+b)}$$
$$=\frac{a-b}{(a-b)(x+a)(x+b)}=\frac{1}{(x+a)(x+b)}.$$

7. $$\frac{a^2+b^2}{(a+b)(a-b)}-\frac{2ab^2}{(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{(a^2+b^2)(a^2+ab+b^2)-2ab^2(a+b)}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{a^4+a^2b^2+a^3b+ab^3+a^2b^2+b^4-2a^2b^2-2ab^3}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{a^4+a^3b-ab^3+b^4}{(a+b)(a^3-b^3)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$

$$=\frac{(a^4+a^3b-ab^3+b^4)(a^2-ab+b^2)+2a^2b(a^3-b^3)}{(a+b)(a^3-b^3)(a^2-ab+b^2)}$$

$$=\frac{a^6+2a^2b^4-2ab^5+b^6+2a^5b-2a^2b^4}{(a^3-b^3)(a^3+b^3)}=\frac{a^6+2a^5b-2ab^5+b^6}{a^6-b^6}.$$

8. $$\frac{1}{4(1+x)}-\frac{-1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1-x-(+1+x)}{4(1-x^2)}+\frac{1}{2(1+x^2)}$$

$$=\frac{2}{4(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1}{2(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1+x^2+1-x^2}{2(1-x^4)}$$

$$=\frac{2}{2(1-x^4)}=\frac{1}{1-x^4}.$$

9. $$\frac{1}{(x-y)(y-z)}+\frac{-1}{(x-y)(x-z)}+\frac{1}{(x-z)(y-z)}$$

$$=\frac{(x-z)+(-y+z)+(x-y)}{(x-y)(y-z)(x-z)}=\frac{2(x-y)}{(x-y)(y-z)(x-z)}=\frac{2}{(x-z)(y-z)}.$$

10. $$\frac{1}{a(a-b)(a-c)}+\frac{-1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}$$

$$=\frac{bc(b-c)+ac(-a+c)+ab(a-b)}{abc(a-b)(a-c)(b-c)}$$

$$=\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}=\frac{1}{abc}.$$

LIII.

1. $$\frac{1}{(x+4)(x+5)}+\frac{1}{(x+7)(x+5)}=\frac{x+7+x+4}{(x+4)(x+5)(x+7)}$$

$$=\frac{2x+11}{(x+4)(x+5)(x+7)}.$$

2. $$\frac{1}{(x-7)(x-6)}+\frac{1}{(x-9)(x-6)}=\frac{x-9+x-7}{(x-7)(x-6)(x-9)}$$

$$=\frac{2(x-8)}{(x-6)(x-7)(x-9)}.$$

3. $$\frac{1}{(x+11)(x-4)}+\frac{1}{(x+11)(x-13)}=\frac{x-13+x-4}{(x-4)(x+11)(x-13)}$$

$$=\frac{2x-17}{(x-4)(x+11)(x-13)}.$$

4. $$\frac{1}{(x+1)(x+2)}+\frac{2x}{(x+1)(x+3)}+\frac{1}{(x+2)(x+3)}$$

$$=\frac{x+3+2x(x+2)+x+1}{(x+1)(x+2)(x+3)}=\frac{2x^2+6x+4}{(x+1)(x+2)(x+3)}$$

$$=\frac{2(x^2+3x+2)}{(x^2+3x+2)(x+3)}=\frac{2}{x+3}.$$

5. $$\frac{m^2+mn+2mn}{n(m+n)}-\frac{2mn}{(m+n)^2}=\frac{(m^2+3mn)(m+n)-2mn^2}{n(m+n)^2}$$

$$=\frac{m^3+3m^2n+m^2n+3mn^2-2mn^2}{n(m+n)^2}=\frac{m^3+4m^2n+mn^2}{n(m+n)^2}.$$

6. $$\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}-\frac{2}{1+x^2+x^4}$$

$$=\frac{1+x^3+1-x^3}{1+x^2+x^4}-\frac{2}{1+x^2+x^4}=0.$$

7. $$\frac{5+5x-6+6x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}-\frac{-7x}{3(1-x^2)}=\frac{11x-1+7x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}$$

$$=\frac{(18x-1)(1+x^2)+7x(1-x^2)}{3(1-x^4)}=\frac{18x+18x^3-1-x^2+7x-7x^3}{3(1-x^4)}$$

$$=\frac{11x^3-x^2+25x-1}{3(1-x^4)}.$$

8. $$\frac{(3-x)+2(x-1)}{8(x-1)(3-x)}+\frac{1}{8(x-5)}+\frac{1}{(1-x)(x-3)(x-5)}$$

$$=\frac{(x+1)(x-5)+(x-1)(3-x)}{8(x-1)(3-x)(x-5)}+\frac{1}{(x-1)(3-x)(x-5)}$$

$$=\frac{x^2-4x-5+4x-x^2-3+8}{8(x-1)(3-x)(x-5)}=0.$$

9. $\dfrac{(1-x+x^2-x^3)(1+x)+x^4}{1+x}=\dfrac{1-x^4+x^4}{1+x}=\dfrac{1}{1+x}.$

LIV.

1. Multiply by 2; $x=16$.

2. Multiply by 4; $3x=36$, $x=12$.

3. Multiply by 15; $5x+3x=120$; $8x=120$, $x=15$.

4. Multiply by 28; $7x-4x=84$; $3x=84$, $x=28$.

5. $-\dfrac{4x}{9}=-28$; $\dfrac{4x}{9}=28$, $\dfrac{x}{9}=7$, $x=63$.

6. Multiply by 15; $10x=528-12x$; $22x=528$; $x=24$.

7. Multiply by 12; $8x+48=7x+108$; $x=60$.

8. $\dfrac{2x}{3}+6=\dfrac{4x}{5}$; $10x+90=12x$; $x=45$.

9. $\dfrac{3x}{4}+3=\dfrac{5x}{6}$; $9x+36=10x$; $x=36$.

10. $\dfrac{7x}{8}=\dfrac{9x}{10}-3$; $35x=36x-120$; $x=120$.

11. $\dfrac{5x}{9}+\dfrac{7x}{12}=82$; $20x+21x=82\times36$; $x=72$.

12. $\dfrac{x}{6}+\dfrac{x}{8}=28$; $4x+3x=28\times24$; $x=96$.

13. $\dfrac{5x}{8}-\dfrac{3x}{4}=-8$; $5x-6x=-64$; $x=64$.

14. $9x+360-10x=348$; $-x=-12$; $x=12$.

15. $3x-44=2x-16$; $x=28$.

16. $6x+4x+3x=13$; $13x=13$; $x=1$.

17. Multiply by 70; $14x+28+10x-10=35x-70$; $-11x=-88$; $x=8$.

18. Multiply by 12; $6x+4x=117-3x$; $13x=117$; $x=9$.

19. Multiply by 140; $35x+315+40x=84x-168+420$; $-9x=-63$; $x=7$.

20. Multiply by 105; $357-63x=1015-385x+140x+70$; $182x=728$; $x=4$.

21. Multiply by 7; $2x-10=0$; $2x=10$; $x=5$.

22. Multiply by 329; $141x+188+28x-357=0$; $169x=169$; $x=1$.

23. Multiply by x; $3-3x=1-x$; $-2x=-2$; $x=1$.

24. Multiply by x; $12+x-5x=6$; $-4x=-6$; $x=\frac{3}{2}$.

25. Multiply by 20; $5x+2x+x=800$; $8x=800$; $x=100$.

26. Multiply by 8; $18x+12-4x=29x-348$; $-15x=-360$; $x=24$.

27. Multiply by $100x$; $275x-300=100-325x$; $600x=400$; $x=\frac{2}{3}$.

28. Multiply by 90; $225+540-30x=100x+30+27-18x+36$; $-112x=-672$; $x=6$.

29. Multiply by 12; $4x+3x-10x-144=20x-696$; $-23x=-552$; $x=24$.

30. Multiply by 60; $42x+12-720-45x=36x+156-255x$; $216x=864$; $x=4$.

LV.

1. Multiply by 2; $10x - x - 2 = 142$; $9x = 144$; $x = 16$.

2. Multiply by 3; $3x - 3 + x = 17$; $4x = 20$; $x = 5$.

3. Multiply by 4; $5 - 2x + 8 = 4x - 12x + 16$; $6x = 3$; $x = \frac{1}{2}$.

4. Multiply by 4; $10x - 5x = 9 - 6 + 2x$; $3x = 3$; $x = 1$.

5. Multiply by 30; $60x - 25x + 20 = 210 - 6 + 12x$; $23x = 184$
   $x = 8$.

6. Multiply by 36; $18x + 36 = 56 - 27 - 45x$; $63x = -7$; $x = -\frac{1}{9}$.

7. Multiply by 48; $30x + 18 - 48 + 64x + 24x = 744 - 72 + 40x$;
   $78x = 702$; $x = 9$.

8. Multiply by 385; $55x + 275 - 77x + 154 = 35x + 315$;
   $-57x = -114$; $x = 2$.

9. Multiply by 105; $35x + 35 - 15x + 60 = 21x + 84$; $-x = -11$;
   $x = 11$.

10. Multiply by 24; $24x - 72 - 3x - 6 = 8x$; $13x = 78$; $x = 6$.

11. Multiply by 84; $12x + 60 = 21x + 42 - 28x + 56$; $19x = 38$; $x = 2$.

12. Multiply by 33; $11x - 3x + 3 = 33x - 297$; $-25x = -300$; $x = 12$.

13. Multiply by 70; $14x + 28 = 35x - 70 - 10x + 10$; $-11x = -88$;
    $x = 8$.

14. Multiply by 140; $35x + 315 - 84x + 168 = 420 - 40x$;
    $-9x = -63$; $x = 7$.

15. Multiply by 78; $39x + 39 - 26x + 78 = 6x + 180$; $7x = 63$; $x = 9$.

16. Multiply by 35; $10x - 7x - 21 = 105x - 735$; $-102x = -714$;
    $x = 7$.

17. Multiply by 154 ; $44x + 154 - 126x + 112 = 77x - 847$;
$-159x = -1113$; $x = 7$.

18. Multiply by 572 ; $1001x - 4433 - 176 - 330x = 182x - 208$;
$489x = 4401$; $x = 9$.

19. Multiply by 273 ; $728x - 1365 - 429x + 39 = 147x + 42$;
$152x = 1368$; $x = 9$.

20. Multiply by 56 ; $49x + 63 - 24x - 8 = 126x - 182 - 996 + 36x$;
$-137x = -1233$; $x = 9$.

21. Multiply by 280 ; $28x + 2800x = 140x + 56x + 7x - 400 + 40x +$
26250 ; $2585x = 25850$; $x = 10$.

LVI.

1. $(a+b)x = c$; $x = \dfrac{c}{a+b}$.

2. $5bx - cx = 3c - 2a$; $(5b - c)\,x = 3c - 2a$, etc.

3. $ax + fx = a^2b - bc + d$; $(a + f)\,x = a^2b - bc + d$, etc.

4. $ax - 5x = bc - dm$; $(a - 5)x = bc - dm$, etc.

5. $-a^2x - ax = -a^2b - abc$; $ax + x = ab + bc$; $(a + 1)\,x = b\,(a + c)$, etc.

6. $3acx - 12cdx = abc + 6bcd$; $(3a - 12d)\,x = ab + 6bd$, etc.

7. $3ackx - kx + ackx = -k^2 - 3k - k^2 + 3abk$;
$3acx - x + acx = -k - 3 - k + 3ab$; $(4ac - 1)x = 3ab - 2k - 3$;
$x = \dfrac{3ab - 2k - 3}{4ac - 1}$.

8. $abcx - cmx + ac^2x = ac^2 + abc - mc$;
$abx - mx + acx = ac + ab - m$; $x = 1$.

9. $(a+b)^2 - x^2 = ab + (b-a)x - x^2 - ab$; $x = \dfrac{(a+b)^2}{b-a}$.

10. $a^2 - x^2 = 2a^2 + 2ax - x^2$; $-2ax = a^2$; $x = -\dfrac{a}{2}$.

11. $a^4 + 2a^2x + x^2 = x^2 + 4a^2 + a^4$; $2a^2x = 4a^2$; $x = 2$.

12. $a^4 - x^2 = a^4 + 2ax - x^2$; $-2ax = 0$; $x = 0$.

13. $ax - b + ac = x + ac$; $ax - x = b$; $x = \dfrac{b}{a-1}$.

14. $2ax - 3a + bx = 1$; $2ax + bx = 3a + 1$; $x = \dfrac{3a+1}{2a+b}$.

15. $18a - 4ax + 2b = 3x$; $3x + 4ax = 18a + 2b$; $x = \dfrac{18a+2b}{4a+3}$.

16. $ax^2 - bx - 1 = ax^2 - a$; $-bx = 1 - a$; $x = \dfrac{a-1}{b}$.

17. $mp^2x + mx^3 = mpqx^2 + mx^3$; $mp^2x = mpqx^2$; $p = qx$; $x = \dfrac{p}{q}$.

18. $dx - abd = ac - adx$; $dx + adx = abd + ac$; $x = \dfrac{abd+ac}{ad+d}$.

19. $x^2 - a - ax + x^2 = 2x^2 - ab$; $-ax = a - ab$; $x = b - 1$

20. $3bx - abc + cx^2 = 4bx - abc$; $cx^2 = bx$; $x = \dfrac{b}{c}$.

21. $a^3b + a^2x - b^3 + bx = b^2x - b^3 - a^3b + a^2x$; $bx - b^2x = -2a^3b$;

$x = \dfrac{2a^3}{b-1}$.

22. $6ax - 4b - 3ax + 3a = 6ax - 4b$; $-3ax = -3a$; $x = 1$.

23. $abm^2 - b^2m - amx + bx = 0$; $(b - am)x = bm(b - am)$; $x = bm$.

24. $2a^3b^4 - b^3x + 3a^3bc = 3a^2cx(a+b) - ab^4 + 2a^2b^3x$;

$-b^3x - 3a^3cx - 3a^2bcx - 2a^2b^3x = -3a^3bc - 2a^3b^4 - ab^4$;

$x = \dfrac{3a^3bc + 2a^3b^4 + ab^4}{b^3 + 3a^3c + 3a^2bc + 2a^2b^3}$.

25. $acx^2 + abc - ac^2x + abx - acx^2 = 0$; $ac^2x - abx = abc$; $x = \dfrac{bc}{c^2 - b}$.

26. $ad^2 + ax^2 = acdx + ax^2$; $x = \dfrac{d}{c}$.

27. $ab = (bc + d)\,x + 1$; $x = \dfrac{ab - 1}{bc + d}$.

28. $3ac + cx = 3a^2 + ax + am - mx$; $cx - ax + mx = am - 3ac + 3a^2$;

$$x = \frac{am - 3ac + 3a^2}{c - a + m}.$$

29. $ab + ax + bx + x^2 - ab - ac = \dfrac{a^2c}{b} + x^2$; $ax + bx = ac + \dfrac{a^2c}{b}$;

$$(a + b)x = \frac{(ab + a^2)\,c}{b}\ ;\ x = \frac{ac}{b}.$$

30. $a^2ce - a^2dx - 2abdx - b^2dx - abdx = a^2de - 3abdx$;

$$-a^2dx - b^2dx = a^2de - a^2ce\ ;\quad x = \frac{a^2e\,(c - d)}{(a^2 + b^2)\,d}.$$

LVII.

1. $(3x + 7)(4x + 3) = (3x + 5)(4x + 5)$;
$12x^2 + 37x + 21 = 12x^2 + 35x + 25$; $2x = 4$; $x = 2$.

2. $(x + 6)(2x - 5) = x(2x + 5)$; $2x^2 + 7x - 30 = 2x^2 + 5x$; $2x = 30$; $x = 15$.

3. $(2x + 7)(2x - 1) = (4x - 1)(x + 2)$; $4x^2 + 12x - 7 = 4x^2 + 7x - 2$;
$5x = 5$; $x = 1$.

4. $(5x - 1)(2x - 3) = (5x - 3)(2x + 3)$; $10x^2 - 17x + 3 = 10x^2 + 9x - 9$;

$$-26x = -12\ ;\ x = \frac{6}{13}.$$

5. $4x - 3 + 2(3x - 2) = 0$; $4x - 3 + 6x - 4 = 0$; $10x = 7$; $x = \dfrac{7}{10}$.

6. $2(1-2x)-5(1-5x)=0$; $2-4x-5+25x=0$; $21x=3$; $x=\frac{1}{7}$.

7. $\frac{x+1+x-1}{x^2-1}=\frac{3}{x^2-1}$; $2x=3$; $x=\frac{3}{2}$.

8. $\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{4x+3}{9}$; $\frac{7x-29}{5x-12}=\frac{8x+19-8x-6}{18}$;

$18(7x-29)=13(5x-12)$; $126x-522=65x-156$; $61x=366$,

$x=6$.

9. $\frac{x}{3}-\frac{2}{3}=\frac{x^2-5x}{3x-7}$; $(x-2)(3x-7)=3(x^2-5x)$;

$3x^2-13x+14=3x^2-15x$; $2x=-14$; $x=-7$.

10. $\frac{(3x+2)(x+2)+(2x-4)(x-1)}{x^2+x-2}=5$;

$3x^2+8x+4+2x^2-6x+4=5x^2+5x-10$; $-3x=-18$; $x=6$.

11. Multiply by 210; $35(x+3)-30(11-x)=84(x-4)-10(x-3)$;
$35x+105-330+30x=84x-336-10x+30$; $-9x=-81$;
$x=9$.

12. $(x+1)(2x+2)=2(x-3)(x+6)$; $2x^2+4x+2=2x^2+6x-36$;
$-2x=-38$; $x=19$.

13. $\frac{3x^2(2x+3)+2x+1}{6x^2+3x}=x+1$; $6x^3+9x^2+2x+1=(6x^2+3x)(x+1)$;

$6x^3+9x^2+2x+1=6x^3+9x^2+3x$; $-x=-1$; $x=1$.

14. $\frac{3(x-1)-(x+1)(x+1)}{x^2-1}=\frac{-x^2}{x^2-1}$; $3x-3-x^2-2x-1=-x^2$; $x=4$.

15. $\frac{2(1+x)+8(1-x)}{1-x^2}=\frac{45}{1-x^2}$; $2+2x+8-8x=45$; $-6x=35$;

$x=-\frac{35}{6}$.

16. $\frac{4}{x-8}+\frac{3}{2(x-8)}-\frac{29}{24}=\frac{2}{3(x-8)}$; multiply by $24(x-8)$;

$96+36-29x+232=16$; $-29x=-348$; $x=12$.

17. $x^4-4x^2+20x-24=x^4-4x^2+16x-16$; $4x=8$; $x=2$.

18. $2x^4+2x^3-23x^2+31x=2x^4+2x^3-23x^2+7x+12$; $24x=12$; $x=\frac{1}{2}$.

19. Multiply by $16x$; $4x^2-16x=4x^2-3x-\frac{13}{8}$; $-13x=-\frac{13}{8}$; $x=\frac{1}{8}$

20. $5-x\left(\frac{7x-4}{2x}\right)=\frac{x}{2}-\frac{8x-4}{4}$; $5-\frac{7x-4}{2}=\frac{x}{2}-\frac{4x-2}{2}$;

$10-7x+4=x-4x+2$; $-4x=-12$; $x=3$.

LVIII.

1. $\frac{5x}{10}-2=\frac{25x}{100}+\frac{2x}{10}-1$; $50x-200=25x+20x-100$; $x=20$.

2. $\frac{325x}{100}-\frac{51}{10}+x-\frac{75x}{100}=\frac{39}{10}+\frac{5x}{10}$; $325x-510+100x-75x=390+50x$

$300x=900$; $x=3$.

3. $\frac{125x}{1000}+\frac{x}{100}=13-\frac{2x}{10}+\frac{4}{10}$; $125x+10x=13000-200x+400$;

$335x=13400$; $x=40$.

4. $\frac{3x}{10}+\frac{1305x}{1000}+\frac{5x}{10}=\frac{2295}{100}-\frac{195x}{1000}$;

$300x+1305x+500x=22950-195x$; $2300x=22950$; $x=\frac{459}{46}$.

5. $\frac{2x}{10}-\frac{x}{100}+\frac{5x}{1000}=\frac{117}{10}$; $200x-10x+5x=11700$;

$195x=11700$; $x=60$.

6. $\frac{24x}{10}-\frac{36x-5}{50}=\frac{8x}{10}+\frac{89}{10}$; $120x-36x+5=40x+445$; $44x=440$: $x=10$.

7. $\frac{24x}{10}-\frac{1075}{100}=\frac{25x}{100}$; $240x-1075=25x$; $215x=1075$; $x=5$.

8. $\frac{5x}{10}+2-\frac{75x}{100}=\frac{4x}{10}-11$; $50x+200-75x=40x-1100$;

$-65x=-1300$; $x=20$.

9. $\frac{405}{900x}=\cdot150$; $\frac{45}{100x}=\cdot15$; $45=15x$; $x=3$.

10. $\frac{25x}{10}-\frac{2+x}{7}\times\frac{-7}{4}=\frac{5}{10}-\frac{5x+3}{8}$; $\frac{5x}{2}+\frac{2+x}{4}=\frac{1}{2}-\frac{5x+3}{8}$;

$20x+4+2x=4-5x-3$; $27x=-3$; $x=-\frac{1}{9}$

11. $\frac{85}{20}-\frac{2}{10x}=\frac{17}{4}-\frac{10-x}{10x}$; $85x-4=85x-20+2x$; $-2x=-16$; $x=8$.

12. $\frac{48x}{600}-\frac{30-40x}{2}=1993$; $8x-1500+2000x=199300$;

$2008x=200800$; $x=100$.

13. $\frac{20-30x}{15}+\frac{500x}{125}-\frac{2x-3}{9}=\frac{10x-20}{18}+\frac{25}{9}$;

$\frac{4-6x}{3}+4x-\frac{2x-3}{9}=\frac{5x-10}{9}+\frac{25}{9}$;

$12-18x+36x-2x+3=5x-10+25$; $11x=0$; $x=0$.

14. $\frac{2408}{100x}+\frac{1}{x}\times\frac{4}{100}\times\frac{10x+9}{10}=\frac{2412}{10}$; $\frac{2408}{100x}+\frac{40x+36}{1000x}=\frac{2412}{10}$;

$24080+40x+36=241200x$; $24116=241160x$; $x=\cdot1$.

15. $\frac{5x}{10}+\frac{45x-75}{60}=\frac{12}{2}-\frac{3x-6}{9}$; $\frac{x}{2}+\frac{3x-5}{4}=6-\frac{x-2}{3}$;

$6x+9x-15=72-4x+8$; $19x=95$: $x=5$.

16. $\frac{1}{2}-\frac{35x}{10x-20}-\frac{24-3x}{8}=\frac{375x}{1000}$; $500-\frac{3500x}{x-2}-3000+375x=375x$;

$$-2500=\frac{3500x}{x-2};\ -25x+50=35x;\ 50=60x;\ x=\frac{5}{6}.$$

17. $\frac{15x}{100}+\frac{135x-225}{600}=\frac{36}{20}-\frac{9x-18}{90}$; $\frac{3x}{20}+\frac{45x-75}{200}=\frac{18}{10}-\frac{x-2}{10}$;

$$30x+45x-75=360-20x+40;\ 95x=475;\ x=5.$$

LIX.

1. Let x be the number; then $\frac{x}{2}+\frac{x}{4}+\frac{x}{5}=95$, etc.

2. Let x be the number; then $\frac{x}{12}+\frac{x}{20}+\frac{x}{40}=38$, etc.

3. Let x be the number; then $\frac{x}{4}-\frac{x}{5}=4$, etc.

4. Let x be the number; then $\frac{x}{25}-\frac{x}{35}=8$, etc.

5. Let x be one part; then $60-x$ is the other; and $\frac{x}{7}=\frac{60-x}{8}$, etc.

6. Let x be one part; then $50-x$ is the other;

   and $\frac{x}{4}+\frac{5(50-x)}{6}=40$, etc.

7. Let x be the greater; then $100-x$ is the less; and

   $\frac{x}{4}-\frac{100-x}{3}=11$, etc.

8. Let x be the number; then $x-\left(\frac{x}{3}+\frac{x}{10}+\frac{x}{12}\right)=58$, etc.

9. Let x be the number; then $33-\frac{x}{4}-\frac{x}{5}-\frac{x}{10}=0$, etc.

10. Let x be the number; then $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=\frac{x}{2}+112$, etc.

11. Let x be the number; then $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{12}+30=2x$, etc.

12. Let x be the greater; then $x-8$ is the less; and $\frac{x}{x-8}=3$, etc.

13. Let x be the property in pounds; then $\frac{x}{7}=x-1626$, etc.

14. Let x be the greater; then $x-504$ is the less; and $\frac{x}{x-504}=15$, etc.

15. Let x be the greater; then $5760-x$ is the less; and

$$x-(5760-x)=\frac{x}{3} \text{ etc.}$$

16. Let x be the number; then $x+\frac{x}{2}-60=65-x$, etc.

17. Let x be the greater; then $x-20$ is the less; and $\frac{x}{7}=\frac{x-20}{3}$, etc.

18. Let x be the less; then $31207-x$ is the greater; and

$$\frac{31207-x}{x}=15+\frac{1335}{x}, \text{ etc.}$$

19. Let x be the age of the younger; then $27-x$ is the age of the elder; and $\frac{27-x}{x}=\frac{7}{2}$, etc.

20. Let x be the greater; then $237-x$ is the less; and $237-x=\frac{4x}{5}$, etc.

21. Let x be A's share; then $\frac{2x}{7}$ is B's share; and $x+\frac{2x}{7}=1800$, etc.

22. Let x be one part; then $46-x$ is the other; and $\frac{x}{7}+\frac{46-x}{3}=10$, etc.

23. Let x be one part; then $a-x$ is the other; and $\frac{x}{m}+\frac{a-x}{n}=b$, etc.

24. Let x be one number; then $a-x$ is the other; and $x-(a-x)=b$, etc.

25. Let x be the number; then $\frac{4x}{3}=24$, etc.

26. Let x be B's share; then $\frac{5x}{11}$ is A's share; and $x+\frac{5x}{11}$ is C's share; then $x+\frac{5x}{11}+x+\frac{5x}{11}=864$, etc.

27. Let x be the property; then $\frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600=x$, etc.

28. Let x be the second; then $70-x$ is the first; and $\frac{70-x}{x}=2+\frac{1}{x}$, etc.

29. Let x be the first; then $x+25$ is the second; and $\frac{x+25}{x}=4+\frac{4}{x}$.

30. Let x be the greater; then $208-x$ is the less; and $\frac{x}{4}+\frac{208-x}{3}=4\{x-(208-x)\}-4$, etc.

31. Let x be the number of days; then $\frac{2x}{3}-\frac{x}{2}=13$, etc.

32. Let x be the number of gallons; then $\frac{4x}{5}-10=\frac{2x}{3}$.

33. In 25 years the sum of their ages will be increased by 50 years; hence the sum of their ages at the present time is 50 years. Let x be the age of the father; then $50-x$ is the age of the

son; and the difference of their ages is $x-(50-x)$, or, $2x-50$. In 20 years the age of the father will be $x+20$, and the age of the son $70-x$. Hence the sum in 20 years will be $x+20+(70-x)$, or, 90 years. Therefore, by the question, $2x-50=30$, or, $x=40$.

34. Let x be the number of years; then $70-x=\frac{10}{3}(35-x)$, etc.

35. Let x be the number of years; then $72-x=5(48-x)$, etc.

36. Let x be the number of days; then $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$, etc.

37. Let x be the number of days; then $\frac{1}{50}+\frac{1}{60}+\frac{1}{75}=\frac{1}{x}$, etc.

38. Let x be the number of days; then $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ represents the part done daily by *two* men of A's strength, *two* of B's, and *two* of C's, working together. Hence A, B, and C together do $\frac{1}{2}$ of $\left(\frac{1}{12}+\frac{1}{15}+\frac{1}{20}\right)$ daily.

$$\therefore \frac{1}{x}=\frac{1}{2} \text{ of } \frac{5+4+3}{60}, \text{ or, } x=\frac{120}{12}=10.$$

39. Let x be the number of hours; then $\frac{1}{x}$ is the part A does hourly;

A and B do $\frac{1}{4}$ hourly; A and C do $\frac{5}{18}$ hourly;

$\therefore$ B does $\frac{1}{4}-\frac{1}{x}$, and C does $\frac{5}{18}-\frac{1}{x}$ hourly.

$\therefore$ B and C do $\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$ hourly.

Hence $\frac{7}{36}=\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$, etc.

40. Let x be the number of days; then $\frac{2}{5}+\frac{3}{10}+\frac{4}{15}=\frac{1}{x}$, etc.

41. Let x be the number of days; then B will do the whole in $3x$ days.

$\therefore$ B does $\frac{1}{3x}$ daily. Now A does $\frac{3}{50}$ daily, and in 3 days he does $\frac{9}{50}$, while B in 3 days does $\frac{1}{x}$.

Hence $\frac{9}{50}+\frac{1}{x}=\frac{2}{5}$, etc.

42. Let x be the number of hours; then $\frac{1}{3}+\frac{1}{4}=\frac{1}{x}$, etc.

43. Let x be the number of hours; then $\frac{3}{4}+\frac{3}{10}+\frac{1}{5}=\frac{1}{x}$, etc.

44. Let x be the number of minutes; then $\frac{1}{40}-\frac{1}{60}=\frac{1}{x}$, etc.

45. Let x be the number of minutes; then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$, etc.

46. The first pipe lets in 12 gall. in $\frac{13}{4}$ min., or, $\frac{48}{13}$ gall. in a min.

The second „ $\frac{46}{3}$ gall. in $\frac{5}{2}$ min., or, $\frac{92}{15}$ gall. in a min.

The third „ 17 gall. in 3 min., or, $\frac{17}{3}$ gall. in a min.

$\therefore$ All three together let in $\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)$ gall. in a min.

Let x be the number of minutes in which they fill the vessel;

then $x\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)=755\frac{1}{4}$,

or, $\frac{2160x+3588x+3315x}{585}=\frac{3021}{4}$, or, $\frac{9063x}{585}=\frac{3021}{4}$,

or $\frac{3x}{585}=\frac{1}{4}$; $\therefore$ $x=\frac{585}{12}=48\frac{3}{4}$.

47. Let x be the number of gallons let in per minute by the third pipe; then $x+10$ and $x-4$ are the number of gallons let in by the other pipes; then $15(x+x+10+x-4)=2400$, etc.

48. Let x be the distance from Ely; then $16-x$ is the distance from Cambridge; and $\dfrac{x}{4\frac{4}{9}}=\dfrac{16-x}{\frac{60}{18}}$; $\dfrac{9x}{40}=\dfrac{9(16-x)}{30}$; $3x=64-4x$; $x=9\frac{1}{7}$.

49. Let x be the distance up; then x is the distance down; and $\dfrac{x}{2\frac{1}{3}}+\dfrac{x}{3\frac{1}{2}}=5$; $\dfrac{3x}{7}+\dfrac{2x}{7}=5$; $x=7$; and $\therefore 2x=14$.

50. In 1 hour he walks $\dfrac{a}{b}$ miles; hence in c hours he walks $\dfrac{ac}{b}$ miles.
He walks 1 mile in $\dfrac{b}{a}$ hours; hence he walks d miles in $\dfrac{bd}{a}$ hours.

51. Let x be the number of days; then $286x=244(x+2)$, etc.

52. Let x be the number of hours; then $x\times\dfrac{22\frac{1}{2}}{3}=(x+8)\times\dfrac{31\frac{1}{2}}{5}$; $\dfrac{15x}{2}=\dfrac{63(x+8)}{10}$, etc.

53. Let x be the distance from Cambridge; $60-x$ is the distance from London; then $\dfrac{x}{4}=\dfrac{60-x}{3\frac{3}{4}}$; $\dfrac{x}{4}=\dfrac{240-4x}{15}$, etc.

54. Let x be the number of hours; then $\dfrac{5}{3}\times x=\dfrac{7}{5}(x+8)$; $x=42$.
Hence 50 hours after A started.

55. (1.) Let the time be x minutes past 1; then $x=5+\dfrac{x}{12}+30$, etc.

(2.) Let the time be x minutes past 4; then $x=20+\dfrac{x}{12}+30$, etc.

(3.) Let the time be x minutes past 8; then $x+30=40+\dfrac{x}{12}$, etc.

56. (1.) Let the time be x minutes past 2; then $x=10+\frac{x}{12}+15$, etc.

(2.) Let the time be x minutes past 4; then there are two solutions, (α) $x+15=20+\frac{x}{12}$, etc. (β) $x=20+\frac{x}{12}+15$, etc.

(3.) Let the time be x minutes past 7; then there are two solutions, (α) $x+15=35+\frac{x}{12}$, etc. (β) $x=35+\frac{x}{12}+15$, etc.

57. (1.) Let the time be x minutes past 3; then $x=15+\frac{x}{12}$, etc.

(2.) Let the time be x minutes past 6; then $x=30+\frac{x}{12}$, etc.

(3.) Let the time be x minutes past 9; then $x=45+\frac{x}{12}$, etc.

58. Let $2x$ be the number of apples; he pays for them $\frac{4x}{5}$ pence.

He sells x for $\frac{x}{2}$ pence, and x for $\frac{x}{3}$ pence.

Hence $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

59. Let x be the number of sovereigns he had at first.

Then he has $\frac{x}{2}-\frac{1}{2}$ left after the first gift;

and he has $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}$ left after the second gift.

Then $\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}=0$; $\frac{x-1}{4}=\frac{1}{2}$; $x=3$.

60. $\dfrac{\frac{2}{3}+n}{3n+\frac{69}{3}}=\frac{1}{33}$; $\frac{2+3n}{9n+69}=\frac{1}{33}$; $66+99n=9n+69$, etc.

61. Let x be the number of days; then $x-3$ is the number of days that the pursuing army is in motion; and $18x+25=23(x-3)$, etc.

62. Let x be his original income in pounds;

then $x-\frac{x}{40}-\frac{1}{13}\left(x-\frac{x}{40}\right)=540$, etc.

63. Let x be the original sum; then first remainder is $\frac{x}{2}-50$, or, $\frac{x-100}{2}$;

second remainder is $\frac{4}{5}$ of $\frac{x-100}{2}-30$, or, $\frac{2x-200-150}{5}$;

third remainder is $\frac{3}{4}$ of $\frac{2x-350}{5}-20$, or, $\frac{6x-1050-400}{20}$;

thus $\frac{6x-1450}{20}=10$; $6x=1650$; $x=275$.

64. Let $2x$ be the number of eggs bought; for x I gave $\frac{x}{2}$ pence, and

for x I gave $\frac{x}{3}$ pence; and I sold $2x$ eggs for $\frac{4x}{5}$ pence.

Thus $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

65. Let x be the number of minutes in which C would fill it; then

$\frac{1200}{x}$ = number of gall. let in by C each minute; and therefore

$\frac{1200}{x}+10$ = number of gall. let in by A and B each minute.

Hence $\frac{1200}{x}+10+\frac{1200}{x}=\frac{1200}{24}$; whence $x=60$.

Thus A takes 90 minutes and C 60 minutes.

Also B lets in $\left(\frac{1200}{24}-\frac{1200}{90}-\frac{1200}{60}\right)$ gall. per minute, or,

$50-\frac{40}{3}-20$, or, $\frac{50}{3}$ gall. per minute.

$\therefore$ B takes $1200\div\frac{50}{3}$ minutes, or, 72 minutes to fill it.

66. Let x be the number of days B would take; then $2x$ is the number A would take.

Then $\frac{1}{x}+\frac{1}{2x}$ is the part A and B drink daily,

$\therefore$ $\frac{3}{4}$ of $\left(\frac{1}{x}+\frac{1}{2x}\right)$ is the part C drinks daily.

Then $\frac{1}{x}+\frac{1}{2x}+\frac{3}{4x}+\frac{3}{8x}=\frac{1}{24}$; $x=63$, etc.

67. Let x be the number of shots each fires.

Then $\frac{7x}{12}$ is the number of A's hits, and $\frac{9x}{12}$ of B's hits.

Thus $\frac{7x}{12}+\frac{9x}{12}=32$; $x=24$.

68. Let x be the number of horses; then $2x$ is the number of oxen, and $100-3x$ is the number of sheep. Reducing all the prices to shillings,

$$440x+250\times 2x+30(100-3x)=4700\,;\ x=2, \text{ etc.}$$

69. Let x be the whole number of marks.

Then $\frac{x}{5}$ = number of marks obtained by B.

And $\frac{2x}{5}$ = number of marks assigned to book work in the paper.

Also $\frac{2x}{5}$ = number of marks obtained by A for riders.

$\therefore$ $\frac{2x}{5}+\frac{2x}{5}=160$; $x=200$.

70. Since the man made a mistake of 55 minutes, the distance between the hands was 5 minutes.

Let x be the number of minutes past 2.

Then $x+5=10+\frac{x}{12}$; $x=5\frac{5}{11}$.

71. Let x be the original force; then $\frac{5x}{6}-4000$ = remainder after the defeat.

Hence $\frac{5x}{6} - 1000 =$ number after the reinforcement.

Then $\frac{3}{4}\left(\frac{5x}{6} - 1000\right) = 18000$; $\frac{5x}{8} - 750 = 18000$; $x = 30000$.

72. Let x be the original debt; then $x + \frac{x}{4} - 25000000 =$ debt after the peace.

Then $\frac{4\frac{1}{2} \times x}{100} = \frac{4\left(x + \frac{x}{4} - 25000000\right)}{100}$; $\frac{9x}{2} = 5x - 100000000$, etc.

73. The influx goes on for 10 days and 15 hours, or, 255 hours.
The consumption extends over 10 days (excluding Sunday) of 15 hours each, or, 150 hours.
Hence if x be the number of gallons that come in each hour,

$$255x + 2250 = 150 \times 2x;\ x = 50.$$

LX.

1. $a + x + \frac{3a}{x} = \frac{ax + x^2 + 3a}{x}$.

2. $\frac{a^2 + ax}{x^2} - \frac{2x - 2a}{x} = \frac{a^2 + ax - 2x^2 + 2ax}{x^2} = \frac{a^2 + 3ax - 2x^2}{x^2}$.

3. $\frac{x - y}{x} + \frac{2y}{x - y} = \frac{x^2 - 2xy + y^2 + 2xy}{x(x - y)} = \frac{x^2 + y^2}{x(x - y)}$.

4. $\frac{4a + 4b}{a - b} - \frac{2a^2 - 2b^2}{a^2 + b^2} = \frac{4a^3 + 4ab^2 + 4a^2b + 4b^3 - 2a^3 + 2ab^2 + 2a^2b - 2b^3}{(a - b)(a^2 + b^2)}$, etc.

LXI.

1. $\frac{5x + 4}{35} - \frac{23x}{70} = \frac{10x + 8 - 23x}{70} = \frac{8 - 13x}{70}$.

2. $\frac{\frac{x^2 - y^2}{xy}}{x - y} = \frac{x^2 - y^2}{xy(x - y)} = \frac{x + y}{xy}$.

3. $\dfrac{1-x^2}{\dfrac{x+1}{x}}=\dfrac{x(1-x^2)}{1+x}=x(1-x).$

4. $\dfrac{y\left(\dfrac{x+y}{y}\right)}{x\left(\dfrac{x-y}{x}\right)}=\dfrac{x+y}{x-y}.$

5. $\dfrac{5x^2+x^3+1}{x^2}\div\dfrac{2x^2-x^3+1}{x^2}=\dfrac{5x^2+x^3+1}{2x^2-x^3+1}.$

6. $\dfrac{x^3+1}{x^2}\div\dfrac{x+1}{x}=\dfrac{(x+1)(x^2-x+1)x}{x^2(x+1)}=\dfrac{x^2-x+1}{x}.$

7. $\dfrac{a^3-1}{a^2}\div\dfrac{a-1}{a}=\dfrac{(a-1)(a^2+a+1)a}{a^2(a-1)}=\dfrac{a^2+a+1}{a}.$

8. $\dfrac{x^2-ax+x^2+ax}{x^2-a^2}\div\dfrac{2x}{x^2-a^2}=\dfrac{2x^2}{2x}=x.$

9. $\dfrac{2x}{x^2+x^2}=\dfrac{2x}{2x^2}=\dfrac{1}{x}.$

10. $\dfrac{x^2}{1+x}+1-\dfrac{1}{1+x}=\dfrac{x^2+1+x-1}{1+x}=\dfrac{x(1+x)}{1+x}=x.$

11. $\dfrac{x^2-2xy+y^2+x^2+2xy+y^2}{x^2-y^2}\div\dfrac{x^2-2xy+y^2-x^2-2xy-y^2}{x^2-y^2}$

$=\dfrac{2x^2+2y^2}{-4xy}$, etc.

12. $\dfrac{1+x+x^2}{1}\div\dfrac{x^2+x+1}{x^2}=x^2.$

13. $\dfrac{a^2+2ab+b^2+b^2}{b(a+b)}\div\dfrac{a+b}{ab}=\dfrac{(a^2+2ab+2b^2)ab}{b(a+b)(a+b)}$, etc.

14. $\dfrac{2m^2-3m+1}{m}\div\dfrac{2m-1}{m}=\dfrac{2m^2-3m+1}{2m-1}=m-1.$

15. $\dfrac{c+b+a}{abc}\div\dfrac{(a+b+c)(a-b-c)}{ab}=\dfrac{ab(a+b+c)}{abc(a+b+c)(a-b-c)}$, etc.

LXII.

1. $\dfrac{a^4}{2a^4}+\dfrac{3a^3}{2a^4}+\dfrac{2a^2}{2a^4}+\dfrac{5a}{2a^4}=\dfrac{1}{2}+\dfrac{3}{2a}+\dfrac{1}{a^2}+\dfrac{5}{2a^3}$.

2. $\dfrac{a^2bc}{abcd}+\dfrac{ab^2d}{abcd}+\dfrac{abc^2}{abcd}+\dfrac{bcd^2}{abcd}=\dfrac{a}{d}+\dfrac{b}{c}+\dfrac{c}{d}+\dfrac{d}{a}$.

3. $\dfrac{x^3}{x^2y^2}-\dfrac{3x^2y}{x^2y^2}+\dfrac{3xy^2}{x^2y^2}-\dfrac{y^3}{x^2y^2}=\dfrac{x}{y^2}-\dfrac{3}{y}+\dfrac{3}{x}-\dfrac{y}{x^2}$.

4. $\dfrac{9a^3}{108}-\dfrac{12a^2}{108}+\dfrac{6a}{108}-\dfrac{3}{108}=\dfrac{a^3}{12}-\dfrac{a^2}{9}+\dfrac{a}{18}-\dfrac{1}{36}$.

5. $\dfrac{18p^2}{3pqrs}+\dfrac{12q^2}{3pqrs}-\dfrac{36r^2}{3pqrs}+\dfrac{72s^2}{3pqrs}=\dfrac{6p}{qrs}+\dfrac{4q}{prs}-\dfrac{12r}{pqs}+\dfrac{24s}{pqr}$.

6. $\dfrac{10x^3}{1000}-\dfrac{25x^2}{1000}+\dfrac{75x}{1000}-\dfrac{125}{1000}=\dfrac{x^3}{100}-\dfrac{x^2}{40}+\dfrac{3x}{40}-\dfrac{1}{8}$.

LXIII.

1. $$\begin{array}{l}
1+a)\,2\,(2-2a+2a^2-2a^3+2a^4\ldots \\
\quad\underline{2+2a} \\
\quad\quad -2a \\
\quad\quad \underline{-2a-2a^2} \\
\quad\quad\quad\quad 2a^2 \\
\quad\quad\quad\quad \underline{2a^2+2a^3} \\
\quad\quad\quad\quad\quad -2a^3 \\
\quad\quad\quad\quad\quad \underline{-2a^3-2a^4} \\
\quad\quad\quad\quad\quad\quad\quad 2a^4 \\
\quad\quad\quad\quad\quad\quad\quad \underline{2a^4+2a^5} \\
\quad\quad\quad\quad\quad\quad\quad\quad -2a^5
\end{array}$$

2. $$m+2)\,m\,(1-\frac{2}{m}+\frac{4}{m^2}-\frac{8}{m^3}+\frac{16}{m^4}\cdots$$

$$\underline{m+2}$$
$$-2$$
$$\underline{-2-\frac{4}{m}}$$
$$\frac{4}{m}$$
$$\underline{\frac{4}{m}+\frac{8}{m^2}}$$
$$-\frac{8}{m^2}$$
$$\underline{-\frac{8}{m^2}-\frac{16}{m^3}}$$
$$\frac{16}{m^3}$$

3. $$a+b)\,a-b\,(1-\frac{2b}{a}+\frac{2b^2}{a^2}-\frac{2b^3}{a^3}+\frac{2b^4}{a^4}\cdots$$

$$\underline{a+b}$$
$$-2b$$
$$\underline{-2b-\frac{2b^2}{a}}$$
$$\frac{2b^2}{a}$$
$$\underline{\frac{2b^2}{a}+\frac{2b^3}{a^2}}$$
$$-\frac{2b^3}{a^2}$$
$$\underline{-\frac{2b^3}{a^2}-\frac{2b^4}{a^3}}$$
$$\frac{2b^4}{a^3}$$

4. $$a^2-x^2)\,a^2+x^2\,(1+\frac{2x^2}{a^2}+\frac{2x^4}{a^4}+\frac{2x^6}{a^6}+\frac{2x^8}{a^8}\cdots$$

$$\underline{a^2-x^2}$$
$$2x^2$$
$$\underline{2x^2-\frac{2x^4}{a^2}}$$
$$\frac{2x^4}{a^2}$$
$$\underline{\frac{2x^4}{a^2}-\frac{2x^6}{a^4}}$$
$$\frac{2x^6}{a^4}$$
$$\underline{\frac{2x^6}{a^4}-\frac{2x^8}{a^6}}$$
$$\frac{2x^8}{a^6}$$

5. $$a-x)\,ax\,(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\frac{x^5}{a^4}\cdots$$

$$\underline{ax-x^2}$$
$$x^2$$
$$\underline{x^2-\frac{x^3}{a}}$$
$$\frac{x^3}{a}$$
$$\underline{\frac{x^3}{a}-\frac{x^4}{a^2}}$$
$$\frac{x^4}{a^2}$$
$$\underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}}$$
$$\frac{x^5}{a^3}$$

6. $a+x)\,b\,(\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}\ldots$

$$\begin{array}{l}
b+\frac{bx}{a} \\
\hline
-\frac{bx}{a} \\
-\frac{bx}{a}-\frac{bx^2}{a^2} \\
\hline
\frac{bx^2}{a^2} \\
\frac{bx^2}{a^2}-\frac{bx^3}{a^3} \\
\hline
\frac{bx^3}{a^3} \\
\frac{bx^3}{a^3}-\frac{bx^4}{a^4} \\
\hline
\frac{bx^4}{a^4}
\end{array}$$

7. $1+2x-2x^2)\,1\,(1-2x+6x^2-16x^3+44x^4\ldots$

$$\begin{array}{l}
1+2x-2x^2 \\
\hline
-2x+2x^2 \\
-2x-4x^2+4x^3 \\
\hline
6x^2-4x^3 \\
6x^2+12x^3-12x^4 \\
\hline
-16x^3+12x^4 \\
-16x^3-32x^4+32x^5 \\
\hline
-44x^4-32x^5
\end{array}$$

8. $1-x+x^2)\,1+x\,(1+2x+x^2-x^3-2x^4\ldots$

$$\begin{array}{l}
1-x+x^2 \\
\hline
2x-x^2 \\
2x-2x^2+2x^3 \\
\hline
x^2-2x^3 \\
x^2-x^3+x^4 \\
\hline
-x^3-x^4 \\
-x^3+x^4-x^5 \\
\hline
-2x^4+x^5
\end{array}$$

9. $1-2b)\,1+b\,(1+3b+6b^2+12b^3+24b^4\ldots$

$$\begin{array}{l}
1-2b \\
\hline
3b \\
3b-6b^2 \\
\hline
6b^2 \\
6b^2-12b^3 \\
\hline
12b^3 \\
12b^3-24b^4 \\
\hline
24b^4
\end{array}$$

10. $x+b)\,x^3-b^3\,(x^2-bx+b^2-\dfrac{2b^3}{x}+\dfrac{2b^4}{x^2}\cdots$

$$\underline{x^3+bx^2}$$
$$-bx^2-b^3$$
$$\underline{-bx^2-b^2x}$$
$$b^2x-b^3$$
$$\underline{b^2x+b^3}$$
$$-2b^3$$
$$\underline{-2b^3-\frac{2b^4}{x}}$$
$$\frac{2b^4}{x}$$

11. $x-b)\,a^2\,(\dfrac{a^2}{x}+\dfrac{a^2b}{x^2}+\dfrac{a^2b^2}{x^3}+\dfrac{a^2b^3}{x^4}+\dfrac{a^2b^4}{x^5}\cdots$

$$\underline{a^2-\frac{a^2b}{x}}$$
$$\frac{a^2b}{x}$$
$$\underline{\frac{a^2b}{x}-\frac{a^2b^2}{x^2}}$$
$$\frac{a^2b^2}{x^2}$$
$$\underline{\frac{a^2b^2}{x^2}-\frac{a^2b^3}{x^3}}$$
$$\frac{a^2b^3}{x^3}$$
$$\underline{\frac{a^2b^3}{x^3}-\frac{a^2b^4}{x^4}}$$
$$\frac{a^2b^4}{x^4}$$

12. $a^2+2ax+x^2)\,a^2\,(1-\dfrac{2x}{a}+\dfrac{3x^2}{a^2}-\dfrac{4x^3}{a^3}+\dfrac{5x^4}{a^4}\cdots$

$$\underline{a^2+2ax+x^2}$$
$$-2ax-x^2$$
$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$
$$3x^2+\frac{2x^3}{a}$$
$$\underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}}$$
$$-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$
$$\underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$
$$\frac{5x^4}{a^2}+\frac{4x^5}{a^3}$$

13. Dividend $=$ (Divisor and Quotient) + Remainder

$$=(x-a)(x^2-2ax)+4a^3=x^3-3ax^2+2a^2x+4a^3.$$

14. Dividend $=(m-5)(m^3+5m^2+15m+34)+75$

$$=m^4-10m^2-41m-95.$$

LXIV.

1. $\dfrac{10x^2+15x+6}{30}\times\dfrac{4x+3}{12}=\dfrac{40x^3+90x^2+69x+18}{360}$, etc.

2. $\dfrac{6a^2-5a+10}{30}\times\dfrac{5a-4}{20}=\dfrac{30a^3-49a^2+70a-40}{600}$, etc.

3.
$$\begin{array}{l}
x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\\
x-\dfrac{1}{x}\\
\hline
x^4+x^2+1+\dfrac{1}{x^2}\\
\qquad -x^2-1-\dfrac{1}{x^2}-\dfrac{1}{x^4}\\
\hline
x^4 \qquad\qquad\qquad\quad -\dfrac{1}{x^4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+\dfrac{1}{x^2}\\
x^2+1+\dfrac{1}{x^2}\\
\hline
x^4-x^2+1\\
\qquad +x^2-1+\dfrac{1}{x^2}\\
\qquad\qquad +1-\dfrac{1}{x^2}+\dfrac{1}{x^4}\\
\hline
x^4 \qquad +1 \qquad\quad +\dfrac{1}{x^4}
\end{array}$$

5. $\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)=\dfrac{1}{a^4}-\dfrac{1}{b^4}.$

6. $\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{ab}-\dfrac{1}{b^2}+\dfrac{1}{bc}+\dfrac{1}{ac}-\dfrac{1}{bc}+\dfrac{1}{c^2}=\dfrac{1}{a^2}+\dfrac{2}{ac}-\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

7. $1+\dfrac{b}{a}+\dfrac{b^2}{a^2}-\dfrac{b}{a}-\dfrac{b^2}{a^2}-\dfrac{b^3}{a^3}+\dfrac{b^2}{a^2}+\dfrac{b^3}{a^3}+\dfrac{b^4}{a^4}=1+\dfrac{b^2}{a^2}+\dfrac{b^4}{a^4}.$

8. $1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3$

$$\begin{array}{l}
1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\
1+\frac{1}{2}x+\frac{1}{4}x^2 \\
\hline
1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\
\quad+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{16}x^3-\frac{1}{32}x^4 \\
\qquad\quad+\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{32}x^4-\frac{1}{64}x^5 \\
\hline
1+\frac{x^2}{8}\qquad\quad-\frac{x^3}{8}\quad-\frac{x^5}{64}.
\end{array}$$

9. $\frac{15+18x-14x^2}{6x^2}\times\frac{4-2x-x^2}{2x^2}=\frac{60+42x-107x^2+10x^3+14x^4}{12x^4}$, etc.

10. $\frac{a^4}{b^4}+1+\frac{2a^2}{b^2}-1-\frac{b^4}{a^4}-\frac{2b^2}{a^2}-\frac{2a^2}{b^2}-\frac{2b^2}{a^2}-4=\frac{a^4}{b^4}-\frac{b^4}{a^4}-\frac{4b^2}{a^2}-4.$

LXV.

1. $\frac{x^4-1}{x^2}\div\frac{x^2+1}{x}=\frac{x^2-1}{x}$, etc.

2. $\frac{a^2b^2-1}{b^2}\div\frac{ab-1}{b}=\frac{ab+1}{b}$, etc.

3. $\frac{m^3n^3+1}{n^3}\div\frac{mn+1}{n}=\frac{m^2n^2-mn+1}{n^2}$, etc.

4. $\frac{c^5d^5-1}{d^5}\div\frac{cd-1}{d}=\frac{c^4d^4+c^3d^3+c^2d^2+cd+1}{d^4}$, etc.

5. $\frac{x^4+2x^2y^2+y^4}{x^2y^2}\div\frac{x^2+y^2}{xy}=\frac{x^2+y^2}{xy}$, etc.

6. $\frac{b^4+a^2b^2+a^4}{a^4b^4}\div\frac{b^2-ab+a^2}{a^2b^2}=\frac{b^2+ab+a^2}{a^2b^2}$, etc.

7. $\dfrac{x^6-y^6-3x^4y^2+3x^2y^4}{x^3y^3}\div\dfrac{x^2-y^2}{xy}=\dfrac{x^4-2x^2y^2+y^4}{x^2y^2}$, etc.

8. $\dfrac{6x^5-32x^4+77x^3-86x^2-66x+216}{8}\div\dfrac{x^2-2x+6}{2}$

$=\dfrac{6x^3-10x^2+16x+36}{4}$, etc.

9. $\dfrac{a^6+b^6}{a^3b^3}\div\dfrac{a^2+b^2}{ab}=\dfrac{a^4-a^2b^2+b^4}{a^2b^2}$, etc.

10. $\dfrac{b^3c^3+a^3c^3+a^3b^3-3a^2b^2c^2}{a^3b^3c^3}\div\dfrac{bc+ac+ab}{abc}$

$=\dfrac{b^2c^2-abc^2-ab^2c+a^2c^2-a^2bc+a^2b^2}{a^2b^2c^2}$, etc.

LXVI.

1. $\dfrac{x-3}{10}\times\dfrac{50x+7}{100}=\dfrac{50x^2-143x-21}{1000}$, etc.

2. $\left(\dfrac{5x}{100}+7\right)\times\left(\dfrac{2x}{10}-3\right)=\dfrac{5x+700}{100}\times\dfrac{2x-30}{10}=\dfrac{10x^2+1250x-21000}{1000}$, etc.

3. $\dfrac{3x-2y}{10}\times\dfrac{4x+7y}{10}=\dfrac{12x^2+13xy-14y^2}{100}$, etc.

4. $\dfrac{43x+52y}{10}\times\dfrac{4x-6y}{100}=\dfrac{172x^2-50xy-312y^2}{1000}$, etc.

5. $\cdot000027-\cdot001+\cdot000343+\cdot00063=\cdot001-\cdot001=0.$

6. $\cdot343-\cdot0441+\cdot00189-\cdot000027=\cdot34489-\cdot044127=\cdot300763.$

LXVII.

1. $a_1x\left(\dfrac{a_1x+a_2x^2+a_3x^3+a_4x^4+\dots}{a_1x}\right)=a\,x\left(1+\dfrac{a_2x}{a_1}+\dfrac{a_3x^2}{a_1}+\dfrac{a_4x^3}{a_1}+\dots\right).$

2. $xyz\left(\frac{xy-xz+yz}{xyz}\right)=xyz\left(\frac{1}{z}-\frac{1}{y}+\frac{1}{x}\right).$

3. $x^2\left(\frac{x^2+xy+y^2}{x^2}\right)=x^2\left(1+\frac{y}{x}+\frac{y^2}{x^2}\right).$

4. $(a+b)\left\{\frac{(a+b)^3-c(a+b)^2-d(a+b)+e}{a+b}\right\}$

$$=(a+b)\left\{(a+b)^2-c(a+b)-d+\frac{e}{a+b}\right\}.$$

LXVIII.

1. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}=\lambda$; and $a=\lambda b$, $c=\lambda d$. Then

(1.) $\frac{a-b}{b}=\frac{\lambda b-b}{b}=\lambda-1$

$\frac{c-d}{d}=\frac{\lambda d-d}{d}=\lambda-1.$

(2.) $\frac{a}{a+b}=\frac{\lambda b}{\lambda b+b}=\frac{\lambda}{\lambda+1}$

$\frac{c}{c+d}=\frac{\lambda d}{\lambda d+d}=\frac{\lambda}{\lambda+1}.$

(3.) $\frac{3a}{4a-5b}=\frac{3\lambda b}{4\lambda b-5b}=\frac{3\lambda}{4\lambda-5}$

$\frac{3c}{4c-5d}=\frac{3\lambda d}{4\lambda d-5d}=\frac{3\lambda}{4\lambda-5}.$

(4.) $\frac{a^2+b^2}{a^2-b^2}=\frac{\lambda^2b^2+b^2}{\lambda^2b^2-b^2}=\frac{\lambda^2+1}{\lambda^2-1}$

$\frac{c^2+d^2}{c^2-d^2}=\frac{\lambda^2d^2+d^2}{\lambda^2d^2-d^2}=\frac{\lambda^2+1}{\lambda^2-1}.$

(5.) $\frac{8a+b}{4a+7b}=\frac{8\lambda b+b}{4\lambda b+7b}=\frac{8\lambda+1}{4\lambda+7}$

$\frac{8c+d}{4c+7d}=\frac{8\lambda d+d}{4\lambda d+7d}=\frac{8\lambda+1}{4\lambda+7}.$

(6.) $\dfrac{a^2-b^2}{c^2-d^2}=\dfrac{\lambda^2b^2-b^2}{\lambda^2d^2-d^2}=\dfrac{b^2(\lambda^2-1)}{d^2(\lambda^2-1)}=\dfrac{b^2}{d^2}$

$\dfrac{ab}{cd}=\dfrac{\lambda b\times b}{\lambda d\times d}=\dfrac{b^2}{d^2}.$

(7.) $\dfrac{11a+b}{11c+d}=\dfrac{11\lambda b+b}{11\lambda d+d}=\dfrac{b(11\lambda+1)}{d(11\lambda+1)}=\dfrac{b}{d}$

$\dfrac{13a+b}{13c+d}=\dfrac{13\lambda b+b}{13\lambda d+d}=\dfrac{b(13\lambda+1)}{d(13\lambda+1)}=\dfrac{b}{d}.$

(8.) $\dfrac{a^2-ab+b^2}{a^2+ab+b^2}=\dfrac{\lambda^2b^2-\lambda b^2+b^2}{\lambda^2b^2+\lambda b^2+b^2}=\dfrac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}$

$\dfrac{c^2-cd+d^2}{c^2+cd+d^2}=\dfrac{\lambda^2d^2-\lambda d^2+d^2}{\lambda^2d^2+\lambda d^2+d^2}=\dfrac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}.$

2. Let each of the fractions $=\lambda$

Then $l=a\lambda-b\lambda$; $m=b\lambda-c\lambda$; $n=c\lambda-a\lambda$

And $l+m+n=a\lambda-b\lambda+b\lambda-c\lambda+c\lambda-a\lambda=0.$

3. Let each of the fractions $=\lambda$

Then $a=\lambda b$, $c=\lambda d$, $e=\lambda f$

And $\dfrac{la+mc+ne}{lb+md+nf}=\dfrac{l\lambda b+m\lambda d+n\lambda f}{lb+md+nf}=\lambda=\dfrac{a}{b}.$

4. Let each of the fractions $=\lambda$

Then $a+b=\lambda b$

$b+c=\lambda c$

$c+a=\lambda a$

Adding, $2a+2b+2c=\lambda(a+b+c)$

Hence $\lambda=2$, and $\therefore\ a+b=2b$, or $a=b$

Similarly we can show that $a=c$, and $b=c$.

5. Let each of the fractions $=\lambda$

Then $a_1=\lambda b_1$; $a_2=\lambda b_2$; $a_3=\lambda b_3$

And $\dfrac{2a_1+3a_2+4a_3}{2b_1+3b_2+4b_3}=\dfrac{2\lambda b_1+3\lambda b_2+4\lambda b_3}{2b_1+3b_2+4b_3}=\lambda=\dfrac{a_1}{b_1}.$

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}$ is less than λ; and $\frac{e}{f}$ is less than λ

$\therefore a=\lambda b$; c is less than λd; e is less than λf

$\therefore a+c+e$ is less than $\lambda b+\lambda d+\lambda f$

$\therefore \frac{a+c+e}{b+d+f}$ is less than λ, that is, less than $\frac{a}{b}$

Next let $\frac{e}{f}=\mu$; then $\frac{a}{b}$ and $\frac{c}{d}$ are each greater than μ

$\therefore e=\mu f$; a is greater than μb; c is greater than μd

$\therefore a+c+e$ is greater than $\mu b+\mu d+\mu f$

$\therefore \frac{a+c+e}{b+d+f}$ is greater than μ, that is than $\frac{e}{f}$.

7. Let each fraction $=\lambda$; then $x_1=\lambda y_1$; and $x_2=\lambda y_2$

Then $\frac{4x_1+5y_1}{7x_1+9y_1}=\frac{4\lambda y_1+5y_1}{7\lambda y_1+9y_1}=\frac{4\lambda+5}{7\lambda+9}$

And $\frac{4x_2+5y_2}{7x_2+9y_2}=\frac{4\lambda y_2+5y_2}{7\lambda y_2+9y_2}=\frac{4\lambda+5}{7\lambda+9}$.

8. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{a^2+ab}{c^2+cd}=\frac{\lambda^2b^2+\lambda b^2}{\lambda^2d^2+\lambda d^2}=\frac{b^2(\lambda^2+\lambda)}{d^2(\lambda^2+\lambda)}=\frac{b^2}{d^2}$

And $\frac{ab-b^2}{cd-d^2}=\frac{\lambda b^2-b^2}{\lambda d^2-d^2}=\frac{b^2(\lambda-1)}{d^2(\lambda-1)}=\frac{b^2}{d^2}$.

9. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{7a+b}{3a+5b}=\frac{7\lambda b+b}{3\lambda b+5b}=\frac{7\lambda+1}{3\lambda+5}$

And $\frac{7c+d}{3c+5d}=\frac{7\lambda d+d}{3\lambda d+5d}=\frac{7\lambda+1}{3\lambda+5}$.

10. Since b is greater than a

bc is greater than ac

$ab+bc$ is greater than $ab+ac$

$\frac{ab+bc}{b(b+c)}$ is greater than $\frac{ab+ac}{b(b+c)}$

$\frac{b(a+c)}{b(b+c)}$ is greater than $\frac{a(b+c)}{b(b+c)}$

$\frac{a+c}{b+c}$ is greater than $\frac{a}{b}$.

11. Since b is less than a, we can show, as in the preceding solution that $\dfrac{a+c}{b+c}$ is less than $\dfrac{a}{b}$.

LXIX.

1. $3\times 16+\dfrac{2\times 4\times \dfrac{1}{4}}{1}\quad \dfrac{1}{\dfrac{1}{4}}=48+2-4=46.$

2. $$\begin{array}{l} 7x^2-12x+5)\,14x^3+\ \ 7x^2-56x+35\,(2x \\ \qquad\qquad\qquad\ \underline{14x^3-24x^2+10x\qquad} \\ \qquad\qquad\qquad\qquad\quad 31x^2-66x+35 \\ \qquad\qquad\qquad\qquad\quad\ \underline{\ 7\qquad\qquad\qquad\quad} \\ \qquad\qquad\qquad\qquad\quad 217x^2-462x+245\,(31 \\ \qquad\qquad\qquad\qquad\quad \underline{217x^2-372x+155} \\ \qquad\qquad\qquad\qquad\qquad\qquad -90x+90 \end{array}$$

Hence H. C. F. is $x-1$, and dividing both terms of the first fraction by it, we get $\dfrac{2x^2+3x-5}{7x-5}$

$$\begin{array}{l} a^2+4a-45)\,a^3-39a+70\,(a-4 \\ \qquad\qquad\quad\ \underline{a^3+4a^2-45a\qquad} \\ \qquad\qquad\qquad -4a^2+6a+70 \\ \qquad\qquad\qquad \underline{-4a^2-16a+180} \\ \qquad\qquad\qquad\qquad\quad 22a-110 \end{array}$$

Hence we find $a-5$ to be the H. C. F., and dividing both terms of the fraction by it, we get $\dfrac{a^2+5a-14}{a+9}$.

3. $\dfrac{a^2+2ap+p^2-a^2+2ap-p^2}{a^2-p^2}\div\dfrac{a^2+2ap+p^2+a^2-2ap+p^2}{a^2-p^2}$

$=\dfrac{4ap}{2a^2+2p^2}=\dfrac{2ap}{a^2+p^2}.$

4. $\dfrac{6x^2-4y^2+3z^2}{24}+\dfrac{6y^2-4z^2+3x^2}{24}+\dfrac{6z^2+4x^2+3y^2}{24}=\dfrac{13x^2+5y^2+5z^2}{24}$

$$\frac{13x^2+5y^2+5z^2}{24}-\frac{24z^2-24x^2+12y^2}{24}=\frac{37x^2-7y^2-19z^2}{24}.$$

5. $\dfrac{16+\dfrac{1}{4}-1+4}{16-\dfrac{1}{4}-1+1}=\dfrac{19\frac{1}{4}}{15\frac{3}{4}}=\dfrac{77}{63}=\dfrac{11}{9}.$

6. $\dfrac{15x^2+18ax-14a^2}{6}\times\dfrac{4x^2-2ax-a^2}{2}$

$$=\frac{60x^4+42ax^3-107a^2x^2+10a^3x+14a^4}{12}.$$

7. $\dfrac{(a-b)(a^2+ab+b^2)}{(a-b)^2}=\dfrac{a^2+ab+b^2}{a-b}=a+2b+\dfrac{3b^2}{a-b}$, by division.

8. $\dfrac{x^2-2xy+y^2+2xy}{x(x-y)}+\dfrac{y^3-xy^2}{x(x^2-y^2)}=\dfrac{x^2+y^2}{x(x-y)}+\dfrac{y^3-xy^2}{x(x^2-y^2)}$

$$=\frac{x^3+xy^2+x^2y+y^3+y^3-xy^2}{x(x^2-y^2)}=\frac{x^3+x^2y+2y^3}{x(x^2-y^2)}.$$

9. $5x^2+9x-2)\ 60x^3-17x^2-4x+1\ (12x-25$

$$\begin{array}{r} 60x^3+108x^2-24x \\ \hline -125x^2+20x+1 \\ -125x^2-225x+50 \\ \hline 245x-49 \end{array}$$

and $\dfrac{245x-49}{5x^2+9x-2}=\dfrac{49(5x-1)}{(x+2)(5x-1)}=\dfrac{49}{x+2}.$

10. $x^4-9x^3+7x^2+9x-8)\ x^4+7x^3-9x^2-7x+8\ (1$

$$\begin{array}{r} x^4-9x^3+7x^2+9x-8 \\ \hline 16x^3-16x^2-16x+16 \end{array}$$

Divide by 16; $x^3 - x^2 - x + 1)\, x^4 - 9x^3 + 7x^2 + 9x - 8\, (x - 8$

$$\begin{array}{r} x^4 - x^3 - x^2 + x \\ \hline -8x^2 + 8x^2 + 8x - 8 \\ -8x^3 + 8x^2 + 8x - 8 \\ \hline \end{array}$$

Hence $x^3 - x^2 - x + 1$ is the H. C. F.

11. $$\frac{x^3}{x^4-1} + \frac{1}{1 - x - \dfrac{1-x}{1-x^2-1}} = \frac{x^3}{x^4-1} + \frac{-x^2}{-x^2+x^3-1+x}$$

$$= \frac{x^3}{x^4-1} + \frac{-x^2(x+1)}{x^4-1} = \frac{-x^2}{x^4-1} = \frac{x^2}{1-x^4}.$$

12. $$a+ab+b^2\left(a+ab+\frac{ab^2}{1-b}\right) = a+ab+b^2\left(\frac{a+ab-ab-ab^2+ab^2}{1-b}\right)$$

$$= a+ab+\frac{ab^2}{1-b} = \frac{a+ab-ab-ab^2+ab^2}{1-b} = \frac{a}{1-b}.$$

13. $$\left(l+\frac{1}{l}\right)\left(l-\frac{1}{l}\right)\left(l^2+\frac{1}{l^2}\right) = \left(l^2-\frac{1}{l^2}\right)\left(l^2+\frac{1}{l^2}\right) = l^4 - \frac{1}{l^4}.$$

14. $$\frac{a+b+2}{ab+a+b+1} + \frac{1}{c+1} = \frac{ac+bc+2c+a+b+2+ab+a+b+1}{abc+ab+ac+a+bc+b+c+1}$$

$$= \frac{ab+ac+bc+2a+2b+2c+3}{abc+ab+ac+a+bc+b+c+1}$$

If this fraction $= 1$, then $ab+ac+bc+2a+2b+2c+3$

$= abc+ab+ac+a+bc+b+c+1$;

or, $ab+ac+bc+2a+2b+2c+3-ab-ac-a-bc-b-c-1 = abc$,

or, $a+b+c+2 = abc$.

15. $\dfrac{ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2}{a^2x^2} \div x - a$

$$
\begin{array}{l}
x - a)\; ax^3 - a^2x^2 - abx^2 - b^2x^2 + a^2bx + a^2b^2 \;(ax^2 - abx - b^2x - ab^2 \\
\qquad \underline{ax^3 - a^2x^2} \\
\qquad\qquad - abx^2 - b^2x^2 + a^2bx \\
\qquad\qquad \underline{- abx^2 + a^2bx} \\
\qquad\qquad\qquad - b^2x^2 + a^2b^2 \\
\qquad\qquad\qquad \underline{- b^2x^2 + ab^2x} \\
\qquad\qquad\qquad\qquad - ab^2x + a^2b^2 \\
\qquad\qquad\qquad\qquad \underline{- ab^2x + a^2b^2}
\end{array}
$$

Hence the result is $\dfrac{ax^2 - abx - b^2x - ab^2}{a^2x^2}$, or, $\dfrac{1}{a} - \dfrac{b}{ax} - \dfrac{b^2}{a^2x} - \dfrac{b^2}{ax^2}$.

16. $\left(\dfrac{a}{bc} + \dfrac{b}{ac} + \dfrac{c}{ab}\right) \div \left(\dfrac{b}{ac} + \dfrac{c}{ab} + \dfrac{a}{bc}\right) = 1$

Also, since $s = \dfrac{a+b+c}{2}$

$s - a = \dfrac{b+c-a}{2}$; $s - b = \dfrac{a+c-b}{2}$; $s - c = \dfrac{a+b-c}{2}$.

Hence $\dfrac{s.(s-a) + (s-b)(s-c)}{bc} = \dfrac{\dfrac{(b+c)^2 - a^2}{4} + \dfrac{a^2 - (b-c)^2}{4}}{bc}$

$= \dfrac{b^2 + 2bc + c^2 - a^2 + a^2 - b^2 + 2bc - c^2}{4bc} = \dfrac{4bc}{4bc} = 1.$

17. $\dfrac{1}{1 + \dfrac{x}{a}} + \dfrac{1}{1 - \dfrac{x}{a}} + \dfrac{2}{1 + \dfrac{x^2}{a^2}} = \dfrac{a}{a+x} + \dfrac{a}{a-x} + \dfrac{2a^2}{a^2+x^2}$

$= \dfrac{a^2 - ax + a^2 + ax}{a^2 - x^2} + \dfrac{2a^2}{a^2 + x^2} = \dfrac{2a^2(a^2 + x^2) + 2a^2(a^2 - x^2)}{a^4 - x^4} = \dfrac{4a^4}{a^4 - x^4}.$

18. $$\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}-\frac{2\,(a^2-b^2)}{a^2+b^2}=\frac{2\,(a^2+b^2)}{a^2-b^2}-\frac{2\,(a^2-b^2)}{a^2+b^2}$$

$$=\frac{2\,(a^4+2a^2b^2+b^4)\;-2\,(a^4-2a^2b^2+b^4)}{a^4-b^4}=\frac{8a^2b^2}{a^4-b^4}.$$

19. $$\frac{2ab-a^2-2ab-b^2}{2a\,(a+b)}+\frac{a^2+b^2}{2a\,(a-b)}=\frac{(a-b)\,(-a^2-b^2)\;+\;(a+b)\,(a^2+b^2)}{2a\,(a^2-b^2)}$$

$$=\frac{-a^3-ab^2+a^2b+b^3+a^3+ab^2+a^2b+b^3}{2a\,(a^2-b^2)}=\frac{2a^2b+2b^3}{2a\,(a^2-b^2)}=\frac{b\,(a^2+b^2)}{a\,(a^2-b^2)}.$$

20. $$\frac{a^2-ab+b^2}{(a-b)^3}\times\frac{(a+b)\,(a-b)}{a^2+b^2}=\frac{(a^2-ab+b^2)\,(a+b)}{(a-b)^2\,(a^2+b^2)}=\frac{a^3+b^3}{(a-b)^2\,(a^2+b^2)}.$$

21. $$\frac{2}{(x^2-1)^2}-\frac{1}{2\,(x-1)^2}+\frac{1}{x^2-1}=\frac{2}{(x^2-1)^2}-\frac{(x+1)^2}{2\,(x^2-1)^2}+\frac{x^2-1}{(x^2-1)^2}$$

$$=\frac{4-x^2-2x-1+2x^2-2}{2\,(x^2-1)^2}=\frac{x^2-2x+1}{2\,(x^2-1)^2}=\frac{1}{2\,(x+1)^2}.$$

22. $$\frac{(a+b)^2-c^2}{c^2-(a-b)^2}\div\frac{a+b+c}{b+c-a}=\frac{(a+b+c)\,(a+b-c)\,(b+c-a)}{(c+a-b)\,(c-a+b)\,(a+b+c)}=\frac{a+b-c}{a-b+c}.$$

23. $$\left(\frac{x^2}{x+1}+1-\frac{1}{x+1}\right)\div\left(\frac{x^2}{x-1}-x-\frac{1}{x-1}\right)$$

$$=\frac{x^2+x+1-1}{x+1}\div\frac{x^2-x^2+x-1}{x-1}=\frac{x^2+x}{x+1}\div 1=\frac{x\,(x+1)}{x+1}=x.$$

24. $$\left(\frac{\dfrac{a+b}{2}-a}{\dfrac{a+b}{2}-b}\right)^3-\frac{\dfrac{a+b}{2}-2a+b}{\dfrac{a+b}{2}+a-2b}=\left(\frac{b-a}{a-b}\right)^3-\frac{3b-3a}{3a-3b}=(-1)^3-(-1)$$

$$=-1+1=0.$$

25. $$\frac{(a+b-c)\,(a-b+c)}{(a+c+b)\,(a+c-b)}+\frac{(b+a-c)\,(b-a+c)}{(a+b+c)\,(a+b-c)}+\frac{(c+a-b)\,(c-a+b)}{(b+c+a)\,(b+c-a)}$$

$$=\frac{a+b-c}{a+c+b}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{b+c+a}=\frac{b+c+a}{a+b+c}=1.$$

26. $\dfrac{x(x-4)(x^2-4)(x^2-4)}{x^2(x-2)(x-2)}=\dfrac{(x-4)(x+2)(x+2)}{x}=\dfrac{(x-4)(x+2)^2}{x}.$

27. $\dfrac{(a+1)(a-1)(a^3+1)(a^3-1)}{(a+1)(a+1)a^2(a-1)(a-1)}=\dfrac{(a^2-a+1)(a^2+a+1)}{a^2}=\dfrac{a^4+a^2+1}{a^2}.$

28. $\dfrac{1+x-x^2}{x^3}-\dfrac{1}{(x^2+1)^2}+\dfrac{(x^2+1)(x-1)}{(x^2+1)^2}-\dfrac{3}{x^2(x^2+1)^2}$

$$=\frac{1+x-x^2}{x^3}-\frac{x^3}{x^3(x^2+1)^2}+\frac{x^3(x^3-x^2+x-1)}{x^3(x^2+1)^2}-\frac{3x}{x^3(x^2+1)^2}$$

$$=\frac{(x^4+2x^2+1)(1+x-x^2)-x^3+x^3(x^3-x^2+x-1)-3x}{x^3(x^2+1)^2}$$

$$=\frac{1+x+x^2+2x^3-x^4+x^5-x^6-x^3+x^6-x^5+x^4-x^3-3x}{x^3(x^2+1)^2}$$

$$=\frac{1-2x+x^2}{x^3(x^2+1)^2}=\frac{(x-1)^2}{x^3(x^2+1)^2}.$$

29. $\dfrac{x^6-a^2x^4+a^4x^2-a^6}{a^3x^3}\div\dfrac{x^2-a^2}{ax}=\dfrac{x^4+a^4}{a^2x^2}=\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}.$

30. $\left\{\dfrac{a^2+2ab+b^2-a^2+2ab-b^2}{2(a^2-b^2)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}$

$$=\left\{\frac{2ab+2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}=\frac{2b(a+b)(a-b)}{(a+b)(a-b)2b}=1.$$

31. $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc+b^2-2bc+c^2+c^2-2ac+a^2-2ab+b^2}{a^2+b^2+c^2}$

$$=\frac{3a^2+3b^2+3c^2}{a^2+b^2+c^2}=3.$$

32. $\dfrac{1+3x^2+2x^3}{(3-2x-7x^2)^4}-\dfrac{(1-x-3x^2)(3-2x-7x^2)}{(3-2x-7x^2)^4}$

$$=\frac{1+3x^2+2x^3-(3-5x-14x^2+13x^3+21x^4)}{(3-2x-7x^2)^4}$$

$$=\frac{-2+5x+17x^2-11x^3-21x^4}{(3-2x-7x^2)^4}.$$

33. $$\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{(x^2-y^2)(x^2+y^2)}+\frac{x^2+2xy+y^2-x^2+2xy-y^2}{(x-y)(x+y)}$$
$$=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\times\frac{(x-y)(x+y)}{4xy}=\frac{xy}{x^2+y^2}.$$

34. $$\frac{x^3-y^3}{y^3}\times\frac{y}{x-y}+\frac{x^3-y^3}{y^3}\times\frac{-y^3}{x^2+xy+y^2}=\frac{x+y}{y}+\frac{(x-y)(-1)}{y}=\frac{2y}{y}=2.$$

35. $$\frac{a(a-b)}{(a-b)(a^2+ab+b^2)}\times\frac{a^2+ab+b^2}{a+b}+\frac{a^3-b^3}{a^3+b^3}\times\frac{a^2-ab+b^2}{a^2+ab+b^2}$$
$$=\frac{a}{a+b}+\frac{a-b}{a+b}=\frac{2a-b}{a+b}.$$

36. $$\frac{2-(x-1)}{4(x-1)^2}+\frac{(x-1)^2-4}{4(x-1)^2(x+1)}=\frac{3-x}{4(x-1)^2}+\frac{x^2-2x-3}{4(x-1)^2(x+1)}$$
$$=\frac{3x+3-x^2-x+x^2-2x-3}{4(x-1)^2(x+1)}=0.$$

37. $$\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(a-b)(x-a)+bx}{abx(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$
$$=\frac{(ax+ab-a^2)(x-b)-ax(x-a)}{abx(a-b)(x-a)(x-b)}$$
$$=\frac{ax^2-abx+abx-ab^2-a^2x+a^2b-ax^2+a^2x}{abx(a-b)(x-c)(x-b)}=\frac{1}{x(x-a)(x-b)}.$$

38. $$\frac{s}{a}-\frac{a}{a}+\frac{s}{b}-\frac{b}{b}+\frac{s}{c}-\frac{c}{c}+\ldots\text{to } n \text{ terms}$$
$$=\frac{s}{a}+\frac{s}{b}+\frac{s}{c}+\ldots\text{ to } n \text{ terms}-(1+1+1\ldots\text{ to } n \text{ terms})$$
$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right)-n.$$

39. $$\frac{x^4+x^2y^2-x^2y^2+y^4}{x^4-y^4}\times\frac{(x^2-y^2)^2}{x^4-2x^2y^2+y^4+x^4+2x^2y^2+y^4}$$
$$=\frac{(x^4+y^4)(x^2-y^2)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)2(x^4+y^4)}=\frac{x^2-y^2}{2(x^2+y^2)}.$$

40. $\dfrac{a+x+a-x}{a+x-a+x} \div \dfrac{a^2+x^2+a^2-x^2}{a^2+x^2-a^2+x^2} = \dfrac{2a}{2x} \times \dfrac{2x^2}{2a^2} = \dfrac{x}{a}.$

41. Since $x^3+\dfrac{1}{x^3}=\left(x+\dfrac{1}{x}\right)\left(x^2-1+\dfrac{1}{x^2}\right)$; and $\dfrac{1}{x^2}-x^2=\left(\dfrac{1}{x}+x\right)\left(\dfrac{1}{x}-x\right)$

the quotient is $x^2-1+\dfrac{1}{x^2}-3\left(\dfrac{1}{x}-x\right)+4$;

or, $x^2+3x+3-\dfrac{3}{x}+\dfrac{1}{x^2}.$

The quotient may also be obtained by long division as in p. 133 of the *Algebra*.

42. $\left(\dfrac{s}{s}-\dfrac{a}{s}\right)+\left(\dfrac{s}{s}-\dfrac{b}{s}\right)+\left(\dfrac{s}{s}-\dfrac{c}{s}\right)+\ldots$ to n terms

$=(1+1+1\ldots$ to n terms$)-\dfrac{1}{s}(a+b+c+\ldots$to n terms$)$

$=n-\dfrac{s}{s}=n-1.$

43. $\dfrac{x^2+xy-xy+y^2}{x^2-y^2} \div \dfrac{x^4-x^2y^2+x^2y^2+y^4}{(x^2+y^2)(x^2-y^2)} = \dfrac{x^2+y^2}{x^2-y^2} \times \dfrac{(x^2+y^2)(x^2-y^2)}{x^4+y^4}$

$=\dfrac{(x^2+y^2)^2}{x^4+y^4}.$

44. $\dfrac{\dfrac{x^2+2xy+y^2-2xy}{(x+y)^2}}{\dfrac{x^2-2xy+y^2+2xy}{(x-y)^2}} \div \dfrac{(x-y)^2}{(x+y)^2} = \dfrac{(x^2+y^2)(x-y)^2}{(x+y)^2(x^2+y^2)} \times \dfrac{(x+y)^2}{(x-y)^2} = 1.$

45. $(a+b)(cd-1)=(1-ab)(c+d)$, or $acd-a+bcd-b=c-abc+d-abd$,

or $a+b+c+d=bcd+acd+abd+abc$

$$=abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$\therefore \dfrac{a+b+c+d}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}}=abcd.$

46. $\dfrac{(p+q)^4}{(p-q)^4} \div \dfrac{(p+q)^3}{(p-q)^3} = \dfrac{p+q}{p-q}.$

47. $\dfrac{1-2x}{3(x^2-x+1)}+\dfrac{1}{6(x+1)}+\dfrac{x+1}{2(x^2+1)}$

$$=\frac{(1-2x)(2x+2)+(x^2-x+1)}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{2x+2-4x^2-4x+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{3-3x-3x^2}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{x^2-x^3-x^4+1-x-x^2+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

48. $\dfrac{1}{x+\dfrac{z}{yz+1}}\div\dfrac{y}{xy+1}-\dfrac{1}{y(xyz+x+z)}$

$$=\frac{yz+1}{xyz+x+z}\times\frac{xy+1}{y}-\frac{1}{y(xyz+x+z)}=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}=1.$$

49. $\dfrac{1}{a-x}+\dfrac{x}{(a-x)^2}-\left(\dfrac{1}{a-y}+\dfrac{y}{(a-y)^2}\right)=\dfrac{a}{(a-x)^2}-\dfrac{a}{(a-y)^2}$

And $\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-y)^2(a-x)}=\dfrac{(a-y)-(a-x)}{(a-x)^2(a-y)^2}$

Hence original fraction $=\dfrac{a(a-y)^2-a(a-x)^2}{(a-y)-(a-x)}=a(a-y)+a(a-x)$

$$=2a^2-ax-ay.$$

50. $\dfrac{3}{abc}\div\dfrac{a+b-c}{abc}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{3}{a+b-c}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{a+b+c}{a+b-c}.$

51. $\dfrac{a+\dfrac{b^2}{b+a}}{a-\dfrac{b^2}{b-a}}(a^6-b^6)=\dfrac{(ab+a^2+b^2)(b-a)}{(ab-a^2-b^2)(b+a)}\cdot(a^6-b^6)$

$$=\frac{b^3-a^3}{-(a^3+b^3)}\cdot(a^6-b^6)=\frac{a^3-b^3}{a^3+b^3}\cdot(a^3+b^3)(a^3-b^3)=(a^3-b^3)^2.$$

LXX.

1. $$\left.\begin{array}{l}6x+21y=123\\6x+\ \ 8y=\ \ 84\end{array}\right\}\text{ subtracting, }13y=39,\ y=3,\text{ etc.}$$

2. $$\left.\begin{array}{l}45x+72y=909\\45x+10y=475\end{array}\right\}\text{ subtracting, }62y=434,\ y=7,\text{ etc.}$$

3. $$\left.\begin{array}{l}26x+34y=378\\26x+13y=273\end{array}\right\}\text{ subtracting, }21y=105,\ y=5,\text{ etc.}$$

4. $$\left.\begin{array}{l}14x+9y=156\\14x+4y=116\end{array}\right\}\text{ subtracting, }5y=40,\ y=8,\text{ etc.}$$

5. $$\left.\begin{array}{l}3x+45y=147\\3x+\ \ 7y=\ \ 71\end{array}\right\}\text{ subtracting, }38y=76,\ y=2,\text{ etc.}$$

6. $$\left.\begin{array}{l}105x+133y=924\\105x+\ \ 51y=678\end{array}\right\}\text{ subtracting, }82y=246,\ y=3,\text{ etc.}$$

7. $$\left.\begin{array}{l}6x+\ \ 4y=\ \ 236\\6x+30y=1146\end{array}\right\}\text{ subtracting, }-26y=-910,\ y=35,\text{ etc.}$$

8. $$\left.\begin{array}{l}156x+108y=420\\156x+\ \ 87y=399\end{array}\right\}\text{ subtracting, }21y=21,\ y=1,\text{ etc.}$$

9. $$\left.\begin{array}{l}504x+98y=2310\\504x+56y=2184\end{array}\right\}\text{ subtracting, }42y=126,\ y=3,\text{ etc.}$$

LXXI.

1. $$\left.\begin{array}{l}6x+21y=156\\6x-10y=\ \ 32\end{array}\right\}\text{ subtracting, }31y=124,\ y=4,\text{ etc.}$$

2. $$\left.\begin{array}{l}105x-60y=825\\105x-91y=763\end{array}\right\}\text{ subtracting, }31y=62,\ y=2,\text{ etc.}$$

3. $$\left.\begin{array}{l}x+y=96\\x-y=\ \ 2\end{array}\right\}\text{ subtracting, }2y=94,\ y=47,\text{ etc.}$$

4. $\left.\begin{aligned}28x+63y&=553\\28x-68y&=160\end{aligned}\right\}$ subtracting, $131y=393$, $y=3$, etc.

5. $\left.\begin{aligned}7x+133y&=679\\7x-\ \ 53y&=121\end{aligned}\right\}$ subtracting, $186y=558$, $y=3$, etc.

6. $\left.\begin{aligned}87x-42y&=525\\87x-56y&=497\end{aligned}\right\}$ subtracting, $14y=28$, $y=2$, etc.

7. $\left.\begin{aligned}342x-426y&=1284\\342x-978y&=\ \ 732\end{aligned}\right\}$ subtracting, $552y=552$, $y=1$, etc.

8. $\left.\begin{aligned}516x+\ \ 24y&=3192\\516x-731y&=\ \ 172\end{aligned}\right\}$ subtracting, $755y=3020$, $y=4$, etc.

9. $\left.\begin{aligned}65x+117y&=2444\\65x-\ \ 10y&=\ \ 285\end{aligned}\right\}$ subtracting, $127y=2159$, $y=17$, etc.

LXXII.

1. $\left.\begin{aligned}12x-21y&=\ \ 66\\12x-28y&=-4\end{aligned}\right\}$ subtracting, $7y=70$, $y=10$, etc.

2. $\left.\begin{aligned}9x-\ \ 5y&=\ \ 52\\9x-24y&=-24\end{aligned}\right\}$ subtracting, $19y=76$, $y=4$, etc.

3. $\left.\begin{aligned}51x+\ \ \ 9y&=\ \ 171\\51x-272y&=-391\end{aligned}\right\}$ subtracting, $281y=562$, $y=2$, etc.

4. $\left.\begin{aligned}21x+49y&=\ \ 546\\21x-57y&=-408\end{aligned}\right\}$ subtracting, $106y=954$, $y=9$, etc.

5. $\left.\begin{aligned}35x-21y&=\ \ 28\\35x-60y&=-50\end{aligned}\right\}$ subtracting, $39y=78$, $y=2$, etc.

6. $\left.\begin{aligned}6x+4y&=\ \ 78\\6x-9y&=-39\end{aligned}\right\}$ subtracting, $13y=117$, $y=9$, etc.

7. $\left.\begin{array}{r} 26x - 65y = -273 \\ 26x - 8y = 240 \end{array}\right\}$ subtracting, $-57y = -513$, $y = 9$, etc.

8. $\left.\begin{array}{r} 105x - 135y = -195 \\ 105x - 49y = 63 \end{array}\right\}$ subtracting, $-86y = -258$, $y = 3$, etc.

9. $\left.\begin{array}{r} 21y + 36x = 528 \\ 21y - 133x = 21 \end{array}\right\}$ subtracting, $169x = 507$, $x = 3$, etc.

LXXIII.

1. $\left.\begin{array}{r} 6x + 9y = 24 \\ 6x + 14y = 14 \end{array}\right\}$ subtracting, $-5y = 10$, $y = -2$, etc.

2. $\left.\begin{array}{r} 15x - 6y = 153 \\ 38x - 6y = 360 \end{array}\right\}$ subtracting, $-23x = -207$, $x = 9$, etc.

3. $\left.\begin{array}{r} 6x - 10y = 102 \\ 6x + 21y = 9 \end{array}\right\}$ subtracting, $-31y = 93$, $y = -3$.

4. $\left.\begin{array}{r} 35y - 15x = 695 \\ 35y + 14x = 637 \end{array}\right\}$ subtracting, $-29x = 58$, $x = -2$, etc.

5. $\left.\begin{array}{r} 8x + 18y = 212 \\ 8x + 17y = 198 \end{array}\right\}$ subtracting, $y = 14$, etc.

6. $\left.\begin{array}{r} 18x - 63y = 72 \\ 18x - 8y = -38 \end{array}\right\}$ subtracting, $-55y = 110$, $y = -2$, etc.

7. $\left.\begin{array}{r} 17x + 12y = 59 \\ 57x - 12y = 459 \end{array}\right\}$ adding, $74x = 518$, $x = 7$, etc.

8. $\left.\begin{array}{r} 24x + 9y = 9 \\ 12x + 9y = 3 \end{array}\right\}$ subtracting, $12x = 6$, $x = \frac{1}{2}$, etc.

9. $\left.\begin{array}{r} 238x - 966y = -1442 \\ 238x - 221y = -697 \end{array}\right\}$ subtracting, $-745y = -745$, $y = 1$, etc.

47. $\dfrac{1-2x}{3(x^2-x+1)}+\dfrac{1}{6(x+1)}+\dfrac{x+1}{2(x^2+1)}$

$$=\frac{(1-2x)(2x+2)+(x^2-x+1)}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{2x+2-4x^2-4x+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{3-3x-3x^2}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{x^2-x^3-x^4+1-x-x^2+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

48. $\dfrac{1}{x+\dfrac{z}{yz+1}}\div\dfrac{y}{xy+1}-\dfrac{1}{y(xyz+x+z)}$

$$=\frac{yz+1}{xyz+x+z}\times\frac{xy+1}{y}-\frac{1}{y(xyz+x+z)}=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}=1.$$

49. $\dfrac{1}{a-x}+\dfrac{x}{(a-x)^2}-\left(\dfrac{1}{a-y}+\dfrac{y}{(a-y)^2}\right)=\dfrac{a}{(a-x)^2}-\dfrac{a}{(a-y)^2}$

And $\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-y)^2(a-x)}=\dfrac{(a-y)-(a-x)}{(a-x)^2(a-y)^2}$

Hence original fraction $=\dfrac{a(a-y)^2-a(a-x)^2}{(a-y)-(a-x)}=a(a-y)+a(a-x)$

$$=2a^2-ax-ay.$$

50. $\dfrac{3}{abc}\div\dfrac{a+b-c}{abc}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{3}{a+b-c}-\dfrac{3-a-b-c}{a+b-c}=\dfrac{a+b+c}{a+b-c}.$

51. $\dfrac{a+\dfrac{b^2}{b+a}}{a-\dfrac{b^2}{b-a}}(a^6-b^6)=\dfrac{(ab+a^2+b^2)(b-a)}{(ab-a^2-b^2)(b+a)}\cdot(a^6-b^6)$

$$=\frac{b^3-a^3}{-(a^3+b^3)}\cdot(a^6-b^6)=\frac{a^3-b^3}{a^3+b^3}\cdot(a^3+b^3)(a^3-b^3)=(a^3-b^3)^2.$$

LXX.

1. $\left.\begin{aligned} 6x+21y&=123 \\ 6x+\ 8y&=\ 84 \end{aligned}\right\}$ subtracting, $13y=39$, $y=3$, etc.

2. $\left.\begin{aligned} 45x+72y&=909 \\ 45x+10y&=475 \end{aligned}\right\}$ subtracting, $62y=434$, $y=7$, etc.

3. $\left.\begin{aligned} 26x+34y&=378 \\ 26x+13y&=273 \end{aligned}\right\}$ subtracting, $21y=105$, $y=5$, etc.

4. $\left.\begin{aligned} 14x+9y&=156 \\ 14x+4y&=116 \end{aligned}\right\}$ subtracting, $5y=40$, $y=8$, etc.

5. $\left.\begin{aligned} 3x+45y&=147 \\ 3x+\ 7y&=\ 71 \end{aligned}\right\}$ subtracting, $38y=76$, $y=2$, etc.

6. $\left.\begin{aligned} 105x+133y&=924 \\ 105x+\ 51y&=678 \end{aligned}\right\}$ subtracting, $82y=246$, $y=3$, etc.

7. $\left.\begin{aligned} 6x+\ 4y&=\ 236 \\ 6x+30y&=1146 \end{aligned}\right\}$ subtracting, $-26y=-910$, $y=35$, etc.

8. $\left.\begin{aligned} 156x+108y&=420 \\ 156x+\ 87y&=399 \end{aligned}\right\}$ subtracting, $21y=21$, $y=1$, etc.

9. $\left.\begin{aligned} 504x+98y&=2310 \\ 504x+56y&=2184 \end{aligned}\right\}$ subtracting, $42y=126$, $y=3$, etc.

LXXI.

1. $\left.\begin{aligned} 6x+21y&=156 \\ 6x-10y&=\ 32 \end{aligned}\right\}$ subtracting, $31y=124$, $y=4$, etc.

2. $\left.\begin{aligned} 105x-60y&=825 \\ 105x-91y&=763 \end{aligned}\right\}$ subtracting, $31y=62$, $y=2$, etc.

3. $\left.\begin{aligned} x+y&=96 \\ x-y&=\ 2 \end{aligned}\right\}$ subtracting, $2y=94$, $y=47$, etc.

LXXIV.

1. $\left.\begin{array}{r}3x+2y=42\\2x+3y=48\end{array}\right\},\ \left.\begin{array}{r}6x+4y=\ \ 84\\6x+9y=144\end{array}\right\},\ -5y=-60,\ y=12,$ etc.

2. $\left.\begin{array}{r}30x+y=630\\20y-x=580\end{array}\right\},\ \left.\begin{array}{r}600x+20y=12600\\20y-\ x\ =\ \ \ 580\end{array}\right\},\ 601x=12020,\ x=20,$ etc.

3. $\left.\begin{array}{r}x+49y=1757\\y+49x=2093\end{array}\right\},\ \left.\begin{array}{r}49x+2401y=86093\\49x+\ \ \ y\ =\ \ 2093\end{array}\right\},\ 2400y=84000,\ y=35,$ etc.

4. $\left.\begin{array}{r}x+y+15=30\\x-y+14=19\end{array}\right\},\ \left.\begin{array}{r}x+y=15\\x-y=\ \ 5\end{array}\right\},\ 2x=20,\ x=10,$ etc.

5. $\left.\begin{array}{r}14x+5y\ =\ \ \ \ 826\\39x-14y=-1609\end{array}\right\},\ \left.\begin{array}{r}196x+70y=\ 11564\\195x-70y=-8045\end{array}\right\},$

$391x=3519,\ x=9,$ etc.

6. $\left.\begin{array}{r}6x+9y=150-5y\\8y-6x=\ \ 9x+12\end{array}\right\},\ \left.\begin{array}{r}6x+14y=150\\8y-15x=\ \ 12\end{array}\right\},\ \left.\begin{array}{r}30x+70y=750\\16y-30x=\ \ 24\end{array}\right\},$

$86y=774,$ etc.

7. $\left.\begin{array}{r}7x-y+2=35\\12y-x-10=9\end{array}\right\},\ \left.\begin{array}{r}7x-y=33\\12y-x=19\end{array}\right\},\ \left.\begin{array}{r}7x-\ \ y=\ \ 33\\84y-7x=133\end{array}\right\},\ 83y=166,$ etc.

8. $\left.\begin{array}{l}x+32=2y-48\\12x+12y+20y=30x-15y+2100\end{array}\right\},\ \left.\begin{array}{r}x-\ \ 2y=-80\\47y-18x=2100\end{array}\right\},$

$\left.\begin{array}{r}18x-36y=-1440\\47y-18x=\ \ \ \ 2100\end{array}\right\},$ adding, $11y=660,\ y=60,$ etc.

9. $\left.\begin{array}{r}15x-25y+30=4x+2y\\96-\ \ 3x+6y=6x+4y\end{array}\right\},\ \left.\begin{array}{r}11x-27y=-30\\2y-\ \ 9x=-96\end{array}\right\},$

$\left.\begin{array}{r}99x-243y=-270\\22y-\ \ 99x=-1056\end{array}\right\},$ adding, $-221y=-1326,\ y=6,$ etc.

10. $\left.\begin{array}{l} x+2+24y=\ \ 93 \\ y+5+40x=768 \end{array}\right\}, \left.\begin{array}{l} x+24y=\ \ 91 \\ 40x+y=763 \end{array}\right\}, \left.\begin{array}{l} 40x+960y=3640 \\ 40x+y\qquad\ =\ \ 763 \end{array}\right\},$

$959y=2877$, etc.

11. $\left.\begin{array}{l} 2x-y+21x=14y-42 \\ 6y+18+5y-5x=60x-240 \end{array}\right\}, \left.\begin{array}{l} 23x-15y=-\ \ 42 \\ 11y-65x=-258 \end{array}\right\},$

$\left.\begin{array}{l} 253x-165y=-\ \ 462 \\ 165y-975x=-3870 \end{array}\right\}$ adding, $-722x=-4332$, $x=6$, etc.

12. $\left.\begin{array}{l} 12x-24-200+20x=15y-150 \\ 16y+32=12x+3y+39 \end{array}\right\}, \left.\begin{array}{l} 32x-15y=74 \\ 13y-12x=\ \ 7 \end{array}\right\},$

$\left.\begin{array}{l} 96x-45y=222 \\ 104y-96x=\ \ 56 \end{array}\right\}$, adding, $59y=278$, $y=\frac{278}{59}$, etc.

13. $\left.\begin{array}{l} 5x-6y+39x=52y-26 \\ 10x+12y-9x+6y=24y-24 \end{array}\right\}, \left.\begin{array}{l} 44x-58y=-26 \\ x-\ \ 6y=-24 \end{array}\right\}$

$\left.\begin{array}{l} 22x-29y\ \ =-\ \ 13 \\ 22x-132y=-528 \end{array}\right\}$, $103y=515$, $y=5$, etc.

14. $\left.\begin{array}{l} 15x-9\ -\ 9x+57=24\ -6y+2x \\ 16x+8y-18x+14=12y+36-4x-5y \end{array}\right\}, \left.\begin{array}{l} 4x+6y=-24 \\ 2x+\ \ y=\ \ \ 22 \end{array}\right\},$

$\left.\begin{array}{l} 2x+3y=-12 \\ 2x+\ \ y=\ \ \ 22 \end{array}\right\}$, $2y=-34$, $y=-17$, etc.

15. $\left.\begin{array}{l} 4x+5y=40x-40y \\ 4x-2y+12y=3 \end{array}\right\}, \left.\begin{array}{l} 45y-36x=0 \\ 4x+10y=3 \end{array}\right\}, \left.\begin{array}{l} 5y-\ \ 4x=0 \\ 4x+10y=3 \end{array}\right\}$, $15y=3$, etc.

LXXV.

1. $\left.\begin{array}{l} mpx+npy=ep \\ mpx+mqy=fm \end{array}\right\}$ subtracting $(np-mq)y=ep-fm$, etc.

2. $\left.\begin{array}{l} adx+bdy=cd \\ adx-aey=af \end{array}\right\}$ subtracting $(bd+ae)\,y=cd-af$, etc.

LXXIV.

1. $\left.\begin{array}{l}3x+2y=42\\2x+3y=48\end{array}\right\}$, $\left.\begin{array}{l}6x+4y=84\\6x+9y=144\end{array}\right\}$, $-5y=-60$, $y=12$, etc.

2. $\left.\begin{array}{l}30x+y=630\\20y-x=580\end{array}\right\}$, $\left.\begin{array}{l}600x+20y=12600\\20y-x=580\end{array}\right\}$, $601x=12020$, $x=20$, etc.

3. $\left.\begin{array}{l}x+49y=1757\\y+49x=2093\end{array}\right\}$, $\left.\begin{array}{l}49x+2401y=86093\\49x+y=2093\end{array}\right\}$, $2400y=84000$, $y=35$, etc.

4. $\left.\begin{array}{l}x+y+15=30\\x-y+14=19\end{array}\right\}$, $\left.\begin{array}{l}x+y=15\\x-y=5\end{array}\right\}$, $2x=20$, $x=10$, etc.

5. $\left.\begin{array}{l}14x+5y=826\\39x-14y=-1609\end{array}\right\}$, $\left.\begin{array}{l}196x+70y=11564\\195x-70y=-8045\end{array}\right\}$,

$391x=3519$, $x=9$, etc.

6. $\left.\begin{array}{l}6x+9y=150-5y\\8y-6x=9x+12\end{array}\right\}$, $\left.\begin{array}{l}6x+14y=150\\8y-15x=12\end{array}\right\}$, $\left.\begin{array}{l}30x+70y=750\\16y-30x=24\end{array}\right\}$,

$86y=774$, etc.

7. $\left.\begin{array}{l}7x-y+2=35\\12y-x-10=9\end{array}\right\}$, $\left.\begin{array}{l}7x-y=33\\12y-x=19\end{array}\right\}$, $\left.\begin{array}{l}7x-y=33\\84y-7x=133\end{array}\right\}$, $83y=166$, etc.

8. $\left.\begin{array}{l}x+32=2y-48\\12x+12y+20y=30x-15y+2100\end{array}\right\}$, $\left.\begin{array}{l}x-2y=-80\\47y-18x=2100\end{array}\right\}$,

$\left.\begin{array}{l}18x-36y=-1440\\47y-18x=2100\end{array}\right\}$, adding, $11y=660$, $y=60$, etc.

9. $\left.\begin{array}{l}15x-25y+30=4x+2y\\96-3x+6y=6x+4y\end{array}\right\}$, $\left.\begin{array}{l}11x-27y=-30\\2y-9x=-96\end{array}\right\}$,

$\left.\begin{array}{l}99x-243y=-270\\22y-99x=-1056\end{array}\right\}$, adding, $-221y=-1326$, $y=6$, etc.

o. $\left.\begin{aligned} x+2+24y &= 93 \\ y+5+40x &= 768 \end{aligned}\right\},\ \left.\begin{aligned} x+24y &= 91 \\ 40x+y &= 763 \end{aligned}\right\},\ \left.\begin{aligned} 40x+960y &= 3640 \\ 40x+y &= 763 \end{aligned}\right\},$

$959y=2877$, etc.

1. $\left.\begin{aligned} 2x-y+21x &= 14y-42 \\ 6y+18+5y-5x &= 60x-240 \end{aligned}\right\},\ \left.\begin{aligned} 23x-15y &= -\ 42 \\ 11y-65x &= -258 \end{aligned}\right\},$

$\left.\begin{aligned} 253x-165y &= -\ 462 \\ 165y-975x &= -3870 \end{aligned}\right\}$ adding, $-722x=-4332$, $x=6$, etc.

2. $\left.\begin{aligned} 12x-24-200+20x &= 15y-150 \\ 16y+32 &= 12x+3y+39 \end{aligned}\right\},\ \left.\begin{aligned} 32x-15y &= 74 \\ 13y-12x &= 7 \end{aligned}\right\},$

$\left.\begin{aligned} 96x-45y &= 222 \\ 104y-96x &= 56 \end{aligned}\right\}$, adding, $59y=278$, $y=\dfrac{278}{59}$, etc.

3. $\left.\begin{aligned} 5x-6y+39x &= 52y-26 \\ 10x+12y-9x+6y &= 24y-24 \end{aligned}\right\},\ \left.\begin{aligned} 44x-58y &= -26 \\ x-\ 6y &= -24 \end{aligned}\right\}$

$\left.\begin{aligned} 22x-29y &= -\ 13 \\ 22x-132y &= -528 \end{aligned}\right\}$, $103y=515$, $y=5$, etc.

4. $\left.\begin{aligned} 15x-9\ -\ 9x+57 &= 24\ -6y+2x \\ 16x+8y-18x+14 &= 12y+36-4x-5y \end{aligned}\right\},\ \left.\begin{aligned} 4x+6y &= -24 \\ 2x+\ y &= 22 \end{aligned}\right\},$

$\left.\begin{aligned} 2x+3y &= -12 \\ 2x+\ y &= 22 \end{aligned}\right\}$, $2y=-34$, $y=-17$, etc.

5. $\left.\begin{aligned} 4x+5y &= 40x-40y \\ 4x-2y+12y &= 3 \end{aligned}\right\},\ \left.\begin{aligned} 45y-36x &= 0 \\ 4x+10y &= 3 \end{aligned}\right\},\ \left.\begin{aligned} 5y-\ 4x &= 0 \\ 4x+10y &= 3 \end{aligned}\right\}$, $15y=3$, etc.

LXXV.

1. $\left.\begin{aligned} mpx+\ npy &= ep \\ mpx+mqy &= fm \end{aligned}\right\}$ subtracting $(np-mq)y=ep-fm$, etc.

2. $\left.\begin{aligned} adx+bdy &= cd \\ adx-aey &= af \end{aligned}\right\}$ subtracting $(bd+ae)\,y=cd-af$, etc.

3. $\left.\begin{array}{l} acx - bcy = cm \\ acx + aey = an \end{array}\right\}$, $y(ae + bc) = an - cm$, etc.

4. $\left.\begin{array}{l} cx - dy = 0 \\ cx + cy = ce \end{array}\right\}$ $(c + d)\, y = ce$, etc.

5. $\left.\begin{array}{l} mm'x - m'ny = m'r \\ mm'x + mn'y = mr' \end{array}\right\}$, $(mn' + m'n)\, y = mr' - m'r$, etc.

6. $\left.\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$, $2x = e + b$, $x = \dfrac{a + b}{2}$, etc.

7. $\left.\begin{array}{l} adx + bdy = cd \\ adx + afy = ac^2 \end{array}\right\}$, $(af - bd)\, y = ac^2 - cd$, etc.

8. $\left.\begin{array}{l} abdx + cd^2y = 2d \\ abdx - bcdy = d - b \end{array}\right\}$, $cdy\,(d + b) = d + b$, etc.

9. $\left.\begin{array}{l} 3a^2 + ax = b^2 + by \\ ax + 2by = d \end{array}\right\}$, $\left.\begin{array}{l} ax - by = b^2 - 3a^2 \\ ax + 2by = d \end{array}\right\}$, $3by = 3a^2 - b^2 + d$, etc.

10. $\left.\begin{array}{l} bcx - cy = -2b \\ b^3cy + ac^3 - ab^3 = 2b^4 + bc^4x \end{array}\right\}$, $\left.\begin{array}{l} bc^4x - c^4y \;= -2bc^3 \\ b^3cy - bc^4x = 2b^4 - ac^3 + ab^3 \end{array}\right\}$,

$$b^3cy - c^4y = 2b^4 - ac^3 + ab^3 - 2bc^3,$$

$$y = \frac{2b^4 - ac^3 + ab^3 - 2bc^3}{c\,(b^3 - c^3)} = \frac{2b + a}{c}, \text{ etc.}$$

11. $\left.\begin{array}{l} bx + bc - b^2 + cx + c^2 - bc + ay + a^2 = 2a^2 \\ a^3y = (b + c)^2\,(b - c)\,x \end{array}\right\}$,

$$\left.\begin{array}{l} (b + c)\,x + ay = a^2 + b^2 - c^2 \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\}, \quad \left.\begin{array}{l} a^2(b + c)x + a^3y = a^2(a^2 + b^2 - c^2) \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\},$$

$$(b + c)\,(a^2 + b^2 - c^2)\,x = a^2\,(a^2 + b^2 - c^2)\;;\; x = \frac{a^2}{b + c}, \text{ etc.}$$

12. $\left.\begin{array}{l} 3x + 5y = \dfrac{8b^2m - 2bm^2}{b^2 - m^2} \\ (b^2 - m^2)x + (b + c + m)\,my = b^2m + 2bm^2 + \dfrac{bcm^2}{b + m} \end{array}\right\}$

$$\left.\begin{aligned}&3(b^2-m^2)x+5(b^2-m^2)y=8b^2m-2bm^2\\&3(b^2-m^2)x+3(bm+cm+m^2)y=3b^2m+6bm^2+\frac{3bcm^2}{b+m}\end{aligned}\right\}$$

$$(5b^2-3bm-3cm-8m^2)y=5b^2m-8bm^2-\frac{3bcm^2}{b+m}$$

$$=\frac{5b^3m+5b^2m^2-8b^2m^2-8bm^3-3bcm^2}{b+m}$$

$$=\frac{bm(5b^2-3bm-3cm-8m^2)}{b+m},\text{ etc.}$$

LXXVI.

1. $$\left.\begin{aligned}\frac{4}{x}+\frac{8}{y}=40\\\frac{4}{x}+\frac{3}{y}=20\end{aligned}\right\}\text{ subtracting, }\frac{5}{y}=20\,;\;y=\frac{1}{4},\text{ etc.}$$

2. $$\left.\begin{aligned}\frac{3}{x}+\frac{6}{y}=3a\\\frac{3}{x}+\frac{4}{y}=b\end{aligned}\right\}\text{ subtracting, }\frac{2}{y}=3a-b\,;\;y=\frac{2}{3a-b},\text{ etc.}$$

3. $$\left.\begin{aligned}\frac{ab}{x}+\frac{b^2}{y}=bc\\\frac{ab}{x}+\frac{a^2}{y}=ad\end{aligned}\right\}\text{ subtracting, }\frac{b^2-a^2}{y}=bc-ad\,;\;y=\frac{b^2-a^2}{bc-ad},\text{ etc.}$$

4. $$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}=m\\\frac{a}{x}-\frac{b}{y}=n\end{aligned}\right\}\text{ subtracting, }\frac{2b}{y}=m-n\,;\;y=\frac{2b}{m-n},\text{ etc.}$$

5. $$\left.\begin{aligned}\frac{56}{x}+\frac{40}{y}=152\\\frac{56}{x}-\frac{21}{y}=49\end{aligned}\right\}\text{ subtracting, }\frac{61}{y}=103\,;\;y=\frac{61}{103},\text{ etc.}$$

6. Multiply the second equation by 4, and we get

$$\left.\begin{array}{r}\frac{5}{3x}+\frac{2}{5y}=7\\ \frac{14}{3x}-\frac{2}{5y}=12\end{array}\right\} \text{adding, } \frac{19}{3x}=19;\ x=\frac{1}{3}, \text{ etc.}$$

7. $$\left.\begin{array}{r}\frac{4}{ax}+\frac{6}{by}=10\\ \frac{15}{ax}-\frac{6}{by}=9\end{array}\right\} \text{adding, } \frac{19}{ax}=19;\ x=\frac{1}{a}, \text{ etc.}$$

8. Multiply the first equation by $\frac{n^2}{m}$, and we get

$$\left.\begin{array}{r}\frac{n}{x}+\frac{n^3}{m^2y}=\frac{mn^2+n^3}{m}\\ \frac{n}{x}+\frac{m}{y}=m^2+n^2\end{array}\right\} \text{subtracting, } \frac{n^3-m^3}{m^2y}=\frac{n^3-m^3}{m};\ y=\frac{1}{m}, \text{ et}$$

LXXVII.

1. From the first and second equations we get

$$\left.\begin{array}{r}5x+7y-2z=13\\ 16x+6y+2z=34\end{array}\right\} \text{adding, } 21x+13y=47\ (1)$$

From the first and third equations we get

$$\left.\begin{array}{r}25x+35y-10z=65\\ x-4y+10z=23\end{array}\right\} \text{adding, } 26x+31y=88\ (2)$$

Then from (1) and (2) we find $x=1$, $y=2$; and hence $z=3$.

2. From the first and second equations we get

$$\left.\begin{array}{r}5x+3y-6z=4\\ 9x-3y+6z=24\end{array}\right\} \text{adding, } 14x=28;\ x=2$$

From the second and third we get $2x+y=6$; $\therefore y=2$, etc.

3. From the first and second equations we get

$$\left.\begin{array}{r}15x-9y+6z=63\\ 16x-2y-6z=6\end{array}\right\} \text{adding, } 31x-11y=69\ (1)$$

From the first and third, subtracting, $3x - 6y = -18$ (2)

Then from (1) and (2) we find $x = 4$, $y = 5$; and $\therefore z = 8$.

4. From the first and second equations we get

$$\left.\begin{aligned} 4x - 5y + 2z &= 6 \\ 4x + 6y - 2z &= 40 \end{aligned}\right\} \text{ adding, } 8x + y = 46 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{aligned} 6x + 9y - 3z &= 60 \\ 7x - 4y + 3z &= 35 \end{aligned}\right\} \text{ adding, } 13x + 5y = 95 \text{ (2)}$$

Then from (1) and (2) we find $x = 5$, $y = 6$; and $\therefore z = 8$.

5. From the second and first equations we get

$$\left.\begin{aligned} 5x + 4y + 3z &= 22 \\ 3x + 3y + 3z &= 18 \end{aligned}\right\} \text{ subtracting, } 2x + y = 4 \text{ (1)}$$

From the third and second equations we get

$$\left.\begin{aligned} 15x + 10y + 6z &= 53 \\ 10x + 8y + 6z &= 44 \end{aligned}\right\} \text{ subtracting, } 5x + 2y = 9 \text{ (2)}$$

Then from (1) and (2) we find $x = 1$, $y = 2$; and $\therefore z = 3$.

6. From the first and second equations we get

$$\left.\begin{aligned} 8x + 4y - 3z &= 6 \\ 3x + 9y - 3z &= 21 \end{aligned}\right\} \text{ subtracting, } 5x - 5y = -15 \text{ (1)}$$

From the second and third equations we get

$$\left.\begin{aligned} 4x + 12y - 4z &= 28 \\ 4x - 5y + 4z &= 8 \end{aligned}\right\} \text{ adding, } 8x + 7y = 36 \text{ (2)}$$

Then from (1) and (2) we find $x = 1$, $y = 4$; and $\therefore z = 6$.

7. From the second and first equations we get

$$\left.\begin{aligned} 8x + 4y + 2z &= 50 \\ 2x + 2y + 2z &= 60 \end{aligned}\right\} \text{ subtracting, } 6x + 2y = -10 \text{ (1)}$$

From the third and first equations we get

$$\left.\begin{aligned} 27x+9y+3z&=64 \\ 3x+3y+3z&=90 \end{aligned}\right\} \text{ subtracting, } 24x+6y=-26 \ (2)$$

Then from (1) and (2) we find $x=\frac{2}{3}$, $y=-7$; and $\therefore z=36\frac{1}{3}$

8. From the first and second equations we get

$$\left.\begin{aligned} 4x-3y+z&=9 \\ 27x+3y-15z&=48 \end{aligned}\right\} \text{ adding, } 31x-14z=57 \ (1)$$

From the second and third equations we get

$$\left.\begin{aligned} 36x+4y-20z&=64 \\ x-4y+3z&=2 \end{aligned}\right\} \text{ adding, } 37x-17z=66 \ (2)$$

Then from (1) and (2) we find $x=5$, $z=7$; and $\therefore y=6$.

9. From the first and second equations we get

$$\left.\begin{aligned} 24x+10y-8z&=58 \\ 65x-10y+25z&=290 \end{aligned}\right\} \text{ adding, } 89x+17z=348 \ (1)$$

From the first and third equations we get

$$\left.\begin{aligned} 12x+5y-4z&=29 \\ 85x-5y-5z&=75 \end{aligned}\right\} \text{ adding, } 97x-9z=104 \ (2)$$

Then from (1) and (2) we find $x=2$, $z=10$; and $\therefore y=9$.

10. Subtracting the second equation from the first, we get $2y=20$; $y=10$.

Adding the third equation to the second, we get $2z=10$; $z=5$.

LXXVIII.

1. Let x and y be the numbers; then $x+y=28$; $x-y=4$, etc.
2. Let x and y be the numbers; then $x+y=256$; $x-y=10$, etc.
3. Let x and y be the numbers; then $x+y=13{\cdot}5$; $x-y=1$, etc.

4. Let x and y be the numbers ; then $7x+5y=332$;

$51x-51y=408$, etc.

5. Let x be the age of the father, and y the age of the son.

Then $x-7=4(y-7)$ and $x+7=2(y+7)$, etc.

6. Let x, y, z be the numbers ; then $x+y=70, x+z=80, y+z=90$, etc.

7. Let x and y be the sums contributed by A and B ;

then $400-(x+y)$ is the sum contributed by C.

Then $y=2x+20$, and $400-(x+y)=x+y$, etc.

8. Let x be A's money, and y B's money in shillings.

Then $3(x-10)=y+10$, and $x+10=2(y-10)$, etc.

9. Let x and y be A's and B's shares ; then $760-(x+y)$ is C's share.

Then $x+y=760-x-y+240$; $y+760-x-y=x+360$, etc.

10. Let x and y be the numbers ; then $\frac{x+y}{2}=24$; $\frac{x-y}{2}=17$, etc.

11. Let x be the greater number ; and y the less ; then

$$\frac{x}{y}=4+\frac{3}{y};\ \frac{x+y+38}{x}=2+\frac{2}{x}, \text{ etc.}$$

12. Let x be the first part, y the second, and $\therefore$ $144-(x+y)$ the third.

Then $\frac{x}{y}=3+\frac{2}{y}$; $\frac{144-x-y}{x+y}=2+\frac{6}{x+y}$, etc.

13. Let x be A's money and y B's money in pounds.

Then $x+\frac{y}{2}=120$; $\frac{2x}{3}+y=120$, etc.

14. Let x be the age of the father, y the age of the son.

Then $x-12=y+12$; $x+12=3(y-12)$, etc.

15. Let x be the greater number, and y the less.
Then $3x = 2x + 10$, and $2x + 3y = 24$, etc.

16. Let x be the age of the father, and y the age of the son.
Then $x + y = \frac{1}{2}(x + y + 50)$; $x - y = \frac{1}{3}(x + y + 40)$, etc.

17. Let x be the greater number, and y the less.
Then $\frac{y}{x} = \cdot 21 + \frac{\cdot 0157}{x}$; $\frac{x}{y} = 4 + \frac{\cdot 742}{y}$, etc.

18. Let x be the cost of a barrel of beer, y of a barrel of porter in pounds.
Then $6x + 10y = 51$; $3x + 7y = 32\frac{1}{10}$. Hence we find $x = 3$, and hence 10 barrels of beer can be bought for £30.

19. Let x be the cost of 1 lb. of tea, y of 1 lb. of coffee in pence.
Then $7x + 5y = 352$, and $4x + 9y = 324$, etc.

20. Let x be the cost of a horse, y of a cow in pounds.
Then $12x + 14y = 380$, and $5x + 3y = 130$, etc.

21. Let x be the cost of a yard of silk, y of a yard of cloth in pence.
Then $8x + 19y = 4370$, and $20x + 16y = 6200$, etc.

22. In one day 10 men and 6 women earn £3, 3s.
In one day 4 men and 8 women earn £2, 2s.
Let x be the daily earnings of a man, y of a woman in shillings.
Then $10x + 6y = 63$, and $4x + 8y = 42$, etc.

23. Let x be the number at £37; and y the number at £45.
Then $37x + 45y = 4220$; and $x + y = 100$, etc.

24. Let $10x + y$ be the number.
Then $x + y = 8$; and $10x + y + 36 = 10y + x$, etc.

25. Let $10x + y$ be the number.
Then $x + y = 10$; and $10x + y + 54 = 10y + x$, etc.

26. Let $10x + y$ be the number.
Then $x + y = 9$; and $10x + y + 9 = 10y + x$, etc.

27. Let $10x+y$ be the number.

Then $x+y=6$; and $\dfrac{10x+y}{x+y}=4$, etc.

28. Let $10x+y$ be the number.

Then $x+y=9$; and $\dfrac{10x+y}{x+y}=5$, etc.

29. Let $10x+y$ be the number.

Then $\dfrac{10x+y}{x+y}=7$; and $\dfrac{10y+x-12}{x-y}=9$, etc.

30. Let $10x+y$ be the number.

Then $\dfrac{10x+y}{x+y}=6+\dfrac{3}{x+y}$; and $\dfrac{10y+x}{x+y}=4+\dfrac{9}{x+y}$

or, $10x+y=6x+6y+3$; and $10y+x=4x+4y+9$, etc.

31. Let $10x+y$ be the number.

Then $\dfrac{10x+y}{x+y-2}=5+\dfrac{1}{x+y-2}$; $\dfrac{10y+x}{x+y+2}=5+\dfrac{8}{x+y+2}$, etc.

32. Let $10x+y$ and $10y+x$ be the numbers.

Then $10x+y+9=10y+x$, and $10x+y+10y+x=33$, etc.

33. Let $100x+10x+x$ be the number.

Then $100x+10x+x=37x^2$; $\therefore x=3$, etc.

34. Let $100x+10y+z$ be the number.

Then $y=2z$; $x+z=9$; $x+y+z=17$.

Hence $x=9-z$; and $\therefore 9-z+2z+z=17$; $z=4$, etc.

35. Let $100x+10y+z$ be the number.

Then $x+y+z=21$; $x+y=z+3$;

$100x+10y+z+198=100z+10y+x$, etc.

36. Let $\dfrac{x}{y}$ be the fraction. Then $\dfrac{x+7}{y}=2$, and $\dfrac{x}{y-1}=1$, etc.

37. Let $\dfrac{x}{y}$ be the fraction. Then $\dfrac{x+1}{y}=\dfrac{1}{3}$, and $\dfrac{x}{y+1}=\dfrac{1}{4}$, etc.

38. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+1}=\frac{1}{3}$, etc.

39. Let $\frac{x}{y}$ be the fraction. Then $x+1=y$, and $x=\frac{1}{2}(y+1)$, etc.

40. Let $\frac{x}{y}$ be the fraction. Then $\frac{x-3}{y-3}=\frac{1}{4}$, and $\frac{x+5}{y+5}=\frac{1}{2}$, etc.

41. Let $\frac{x}{y}$ be the fraction. Then $\frac{x}{y+4}=\frac{7}{9}$, and $\frac{x-15}{y}=\frac{20}{41}$, etc.

42. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+17}=\frac{1}{3}$, etc.

43. Let x and y be the sums invested.

Then $\frac{x\times 5}{100}=$ income on first investment.

and $\frac{y\times 4}{100}=$ income on second investment.

$\therefore \frac{x\times 5}{100}=\frac{y\times 4}{100}+10$; and $x+y=2000$, etc.

44. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 10}{100\times 12}=5250$

and $x+\frac{x\times y\times 18}{100\times 12}=5450.$

Hence $\left.\begin{aligned}120x+\ \ xy&=\ 630000\\ \text{and}\quad 200x+3xy&=1090000\end{aligned}\right\}$, $\left.\begin{aligned}360x+3xy&=1890000\\ 200x+3xy&=1090000\end{aligned}\right\}$,

$160x=800000$, etc.

45. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 6}{100}=5200.$

and $x+\frac{x\times y\times 10}{100}=6000$

Hence $\left.\begin{aligned}50x+3xy&=260000\\ 10x+\ \ xy&=\ 60000\end{aligned}\right\}$, $\left.\begin{aligned}50x+3xy&=260000\\ 30x+3xy&=180000\end{aligned}\right\}$,

$20x=80000$, etc.

6. Let x be the number of quarts of the first, y of the second.

Then $\left.\begin{array}{r}x+\ y\ =50\\ 36x+20y=30\times 50\end{array}\right\}$, $\left.\begin{array}{r}36x+36y=1800\\ 36x+20y=1500\end{array}\right\}$, $16y=300$, etc.

7. Let x be the number of lbs. of the cheaper, y of the dearer.

Then $x+y=30$
and $20(x+y)=14x+18y+10x$, etc.

8. Let x be the rate of the rowing in miles an hour in still water, y the rate of the stream in miles an hour.

Then $x+y=12$, $x-y=6$, etc.

9. Let x be the distance in miles, y the rate in miles an hour of pulling.

Then $x=1\frac{2}{3}(y+4)$
and $x-3=4\frac{1}{4}(y-4)$, etc.

10. Let x be the number of leaps the hare takes, and y the length of each in feet.

Then $\frac{5x}{6}$ is the number of leaps the dog takes, and $\frac{9y}{7}$ the length of each in feet.

Then $50y+xy=\frac{5x}{6}\times\frac{9y}{7}$

Divide by y; then $50+x=\frac{15x}{14}$; $x=700$.

11. Let x be the number of leaps the dog takes, and y the length of each in feet.

Then $\frac{4x}{3}$ is the number of leaps the hare takes, and $\frac{2y}{3}$ is the length of each in feet.

Then $50y+\frac{4x}{3}\times\frac{2y}{3}=xy$.

Divide by y; then $50+\frac{8x}{9}=x$; $x=450$, etc.

52. Let x be the number of apples, and y the number of pears.

Then $\frac{x}{4}+\frac{y}{5}=30$; and $\frac{x}{8}+\frac{y}{15}=13$, etc.

53. Let x be the number of men, y the reckoning of each in shillings.

$$\left.\begin{array}{l}\text{Then } (x+3)(y-1)=xy \\ \text{and } (x-2)(y+1)=xy\end{array}\right\}, \quad \left.\begin{array}{l} xy+3y-x-3=xy \\ xy-2y+x-2=xy\end{array}\right\},$$

$$\left.\begin{array}{l}3y-x=3 \\ x-2y=2\end{array}\right\} \text{adding, } y=5, \text{ and hence } x=12.$$

54. Let x be the number that voted for A and C,

y " " for A and B,

z " " for A only.

$$\left.\begin{array}{r}\text{Then } x+\ y+\ z=1056 \\ x+85+\ 98=933 \\ y+85+744=987\end{array}\right\}, \text{ hence } x=750,\ y=158,\ z=148.$$

55. Let x be the distance in miles; y the rate in miles an hour.

$$\left.\begin{array}{l}\text{Then } \frac{x}{y+\frac{1}{2}}=\frac{x}{y}-1\frac{1}{2} \\ \text{and } \frac{x}{y-\frac{1}{2}}=\frac{x}{y}+2\frac{1}{2}\end{array}\right\}, \text{ or, } \left.\begin{array}{l}\frac{2x}{2y+1}=\frac{2x-3y}{2y} \\ \frac{2x}{2y-1}=\frac{2x+5y}{2y}\end{array}\right\}$$

$$\left.\begin{array}{l}4xy=4xy+2x-\ 6y^2-3y \\ 4xy=4xy-2x+10y^2-5y\end{array}\right\}, \quad \left.\begin{array}{l}2x=\ 6y^2+3y \\ 2x=10y^2-5y\end{array}\right\}$$

hence $6y^2+3y=10y^2-5y$; $8y=4y^2$; $8=4y$; $y=2$, etc.

56. Suppose the first crew pulls x strokes of y yards each in a minute.

Then the second crew pulls $\frac{8x}{9}$ strokes of $\frac{90y}{79}$ yards each in a minute.

Hence in one minute the first crew pulls over xy yards,

and in one minute the second crew pulls over $\frac{80xy}{79}$ yards;

$\therefore$ the second is the faster crew.

Again, the second crew gains $\frac{xy}{79}$ yards in a minute;

$\therefore$ it gains $\frac{xy}{79}\div\frac{8x}{9}$ yards in a stroke, or, $\frac{9y}{79\times 8}$ yards.

Now it has to gain $\frac{4\times 90y}{79}$ yards;

$\therefore$ it must take $\frac{4\times 90y}{79}\div\frac{9y}{79\times 8}$ strokes, or, 320 strokes.

57, Let x be the rate of the sculler; y the rate of the barges.

Then $\frac{b}{x}=$ time he takes to meet the first barge

$\frac{a-b}{y}=$ time the first barge takes to meet him.

Hence $\frac{b}{x}=\frac{a-b}{y}$, or, $\frac{b}{a-b}=\frac{x}{y}$. (1)

Also $\frac{b'}{x}=$ time he takes to overtake the second barge

$\frac{b'-a}{y}=$ time the second barge is in motion before he overtakes it;

$\therefore\ \frac{b'}{x}=\frac{b'-a}{y}$, or, $\frac{b'}{b'-a}=\frac{x}{y}$ (2)

From (1) and (2) $\frac{b}{a-b}=\frac{b'}{b'-a}$, or, $bb'-ab=ab'-bb'$,

or $2bb'=ab+ab'$, or, $\frac{2bb'}{abb'}=\frac{ab}{abb'}+\frac{ab'}{abb'}$, or, $\frac{2}{a}=\frac{1}{b'}+\frac{1}{b}$.

58. Let x be the number of feet passed over by the longer train in a second,

y the number of feet passed over by the shorter train in a second.

Then $\frac{3x}{2}+\frac{3y}{2}=176$, and $6x-6y=176$.

Hence $x=\frac{220}{3}$ and $y=44$.

Hence the longer train goes in miles per hour $\frac{220\times 60\times 60}{3\times 3\times 1760}$, or, 50.

and the shorter train goes in miles per hour $\frac{44\times 60\times 60}{3\times 1760}$, or, 30.

59. Let x and y be the circumferences of the wheels in yards.

Then $\frac{120}{x}=\frac{120}{y}+6$, and $\frac{120}{x+\frac{x}{4}}=\frac{120}{y+\frac{y}{5}}+4$;

hence $\frac{20}{x}-\frac{20}{y}=1$, and $\frac{96}{x}-\frac{100}{y}=4$; $\therefore x=4$, $y=5$.

60. Let x be the number of hours he takes to go.

Then $10-x$ is the number of hours he takes to return,

and $x=\frac{2}{3}(10-x)$; whence $x=4$, and $10-x=6$.

Again let y be the number of miles per hour he can row in still water, and z the number of miles per hour the stream flows.

Then $y+z=5$, and $y-z=\frac{20}{6}$; whence $z=\frac{5}{6}$.

61. Let the digits commencing with the left be 1, a, b, c, d, e.

Then the number is $100000+10000a+1000b+100c+10d+e$.

When the 1 is removed to the unit's place

the new number is $100000a+10000b+1000c+100d+10e+1$.

Then $100000a+10000b+1000c+100d+10e+1$

$=300000+30000a+3000b+300c+30d+3e$

$\therefore\ 70000a+7000b+700c+70d+7e=299999$

or, $10000a+1000b+100c+10d+e=42857$

$\therefore\ 100000+10000a+1000b+100c+10d+e=142857$

that is, the required number is 142857.

LXXX.

1.
$$\begin{array}{r|l}
 & 4a^2+12ab+9b^2\ (2a+3b \\
 & 4a^2 \\ \hline
4a+3b & \quad 12ab+9b^2 \\
 & \quad 12ab+9b^2 \\ \hline
\end{array}$$

2.
$$\begin{array}{r|l}
 & 16k^{10}-24k^5l^3+9l^6\ (4k^5-3l^3 \\
 & 16k^{10} \\ \hline
8k^5-3l^3 & \quad -24k^5l^3+9l^6 \\
 & \quad -24k^5l^3+9l^6 \\ \hline
\end{array}$$

3.

$$\begin{array}{r|l} & a^2b^2 + 162ab + 6561 \ (ab + 81 \\ & a^2b^2 \\ \hline 2ab + 81 & \quad 162ab + 6561 \\ & \quad 162ab + 6561 \\ \hline \end{array}$$

4.

$$\begin{array}{r|l} & y^6 - 38y^3 + 361 \ (y^3 - 19 \\ & y^6 \\ \hline 2y^3 - 19 & \quad -38y^3 + 361 \\ & \quad -38y^3 + 361 \\ \hline \end{array}$$

5.

$$\begin{array}{r|l} & 9a^2b^2c^2 - 102abc + 289 \ (3abc - 17 \\ & 9a^2b^2c^2 \\ \hline 6abc - 17 & \quad -102abc + 289 \\ & \quad -102abc + 289 \\ \hline \end{array}$$

6.

$$\begin{array}{r|l} & x^4 - 6x^3 + 19x^2 - 30x + 25 \ (x^2 - 3x + 5 \\ & x^4 \\ \hline 2x^2 - 3x & -6x^3 + 19x^2 \\ & -6x^3 + \ \ 9x^2 \\ \hline 2x^2 - 6x + 5 & \qquad 10x^2 - 30x + 25 \\ & \qquad 10x^2 - 30x + 25 \\ \hline \end{array}$$

7.

$$\begin{array}{r|l} & 9x^4 + 12x^3 + 10x^2 + 4x + 1 \ (3x^2 + 2x + 1 \\ & 9x^4 \\ \hline 6x^2 + 2x & \quad 12x^3 + 10x^2 \\ & \quad 12x^3 + \ \ 4x^2 \\ \hline 6x^2 + 4x + 1 & \qquad 6x^2 + 4x + 1 \\ & \qquad 6x^2 + 4x + 1 \\ \hline \end{array}$$

8.

$$\begin{array}{r|l} & 4r^4 - 12r^3 + 13r^2 - 6r + 1 \ (2r^2 - 3r + 1 \\ & 4r^4 \\ \hline 4r^2 - 3r & -12r^3 + 13r^2 \\ & -12r^3 + \ \ 9r^2 \\ \hline 4r^2 - 6r + 1 & \qquad 4r^2 - 6r + 1 \\ & \qquad 4r^2 - 6r + 1 \\ \hline \end{array}$$

9. $$4n^4 + 4n^3 - 7n^2 - 4n + 4 \; (2n^2 + n - 2$$
$$4n^4$$
$$4n^2 + n \qquad 4n^3 - 7n^2$$
$$4n^3 + n^2$$
$$4n^2 + 2n - 2 \qquad -8n^2 - 4n + 4$$
$$-8n^2 - 4n + 4$$

10. $$1 - 6x + 13x^2 + 12x^3 + 4x4 \; (1 - 3x + 2x^2$$
$$1$$
$$2 - 3x \qquad -6x + 13x^2$$
$$-6x + 9x^2$$
$$2 - 6x + 2x^2 \qquad 4x^2 + 12x^3 + 4x^4$$
$$4x^4 - 12x^3 + 4x^4$$

11. $$x^6 - 4x^5 + 10x^4 - 12x^3 + 9x^2 \; (x^3 - 2x^2 + 3x$$
$$x6$$
$$2x^3 - 2x^2 \qquad -4x^5 + 10x^4$$
$$-4x^5 + 4x^4$$
$$2x^3 - 4x^2 + 3x \qquad 6x^4 - 12x^3 + 9x^2$$
$$6x^4 - 12x^3 + 9x^2$$

12. $$4y^4 - 12y^3z + 25y^2z^2 - 24yz^3 + 16z^4 \; (2y^2 - 3yz + 4z$$
$$4y^4$$
$$4y^2 - 3yz \qquad -12y^3z + 25y^2z^2$$
$$-12y^3z + 9y^2z^2$$
$$4y^2 - 6yz + 4z^2 \qquad 16y^2z^2 - 24yz^3 + 16z^4$$
$$16y^2z^2 - 24yz^3 + 16z^4$$

13. $$a^2 + 4ab + 4b^2 + 9c^2 + 6ac + 12bc\,(a + 2b + 3c$$
$$a^2$$

$$\begin{array}{ll} 2a + 2b & 4ab + 4b^2 \\ & 4ab + 4b^2 \\ 2a + 4b + 3c & 6ac + 12bc + 9c^2 \\ & 6ac + 12bc + 9c^2 \end{array}$$

14. $$a^6 + 2a^5b + 3a^4b^2 + 4a^3b^3 + 3a^2b^4 + 2ab^5 + b^6\,(a^3 + a^2b + ab^2 + b^3$$
$$a^6$$

$$\begin{array}{ll} 2a^3 + a^2b & 2a^5b + 3a^4b^2 \\ & 2a^5b + a^4b^2 \\ 2a^3 + 2a^2b + ab^2 & 2a^4b^2 + 4a^3b^3 + 3a^2b^4 \\ & 2a^4b^2 + 2a^3b^3 + a^2b^4 \\ 2a^3 + 2a^2b + 2ab^2 + b^3 & 2a^3b^3 + 2a^2b^4 + 2ab^5 + b^6 \\ & 2a^3b^3 + 2a^2b^4 + 2ab^5 + b^6 \end{array}$$

15. $$x^6 - 4x^5 + 6x^3 + 8x^2 + 4x + 1\,(x^3 - 2x^2 - 2x - 1$$
$$x^6$$

$$\begin{array}{ll} 2x^3 - 2x^2 & -4x^5 + 6x^3 \\ & -4x^5 + 4x^4 \\ 2x^3 - 4x^2 - 2x & -4x^4 + 6x^3 + 8x^2 \\ & -4x^4 + 8x^3 + 4x^2 \\ 2x^3 - 4x^2 - 4x - 1 & -2x^3 + 4x^2 + 4x + 1 \\ & -2x^3 + 4x^2 + 4x + 1 \end{array}$$

16. $$4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4\,(2x^2 + 2ax + 4b$$
$$4x^4$$

$$\begin{array}{ll} 4x^2 + 2ax & 8ax^3 + 4a^2x^2 \\ & 8ax^3 + 4a^2x^2 \\ 4x^2 + 4ax + 4b^2 & 16b^2x^2 + 16ab^2x + 16b^4 \\ & 16b^2x^2 + 16ab^2x + 16b^4 \end{array}$$

17.

$$9-24x+58x^2-116x^3+129x^4-140x^5+100x^6\,(3-4x+7x^2-10x^3$$

$$\begin{array}{ll}
 & 9 \\
6-4x & -24x+58x^2 \\
 & -24x+16x^2 \\
6-8x+7x^2 & \qquad 42x^2-116x^3+129x^4 \\
 & \qquad 42x^2-56x^3+49x^4 \\
6-8x+14x^2-10x^3 & \qquad\qquad -60x^3+80x^4-140x^5+100x^6 \\
 & \qquad\qquad -60x^3+80x^4-140x^5+100x^6
\end{array}$$

18.

$$16a^4-40a^3b+25a^2b^2-80ab^2x+64b^2x^2+64a^2bx\,(4a^2-5ab+8bx$$

$$\begin{array}{ll}
 & 16a^4 \\
8a^2-5ab & -40a^3b+25a^2b^2 \\
 & -40a^3b+25a^2b^2 \\
8a^2-10ab+8bx & \qquad 64a^2bx-80ab^2x+64b^2x^2 \\
 & \qquad 64a^2bx-80ab^2x+64b^2x^2
\end{array}$$

19.

$$9a^4-24a^3p^3-30a^2t+16a^2p^6+40ap^3t+25t^2\,(3a^2-4ap^3-5t$$

$$\begin{array}{ll}
 & 9a^4 \\
6a^2-4ap^3 & -24a^3p^3+16a^2p^6 \\
 & -24a^3p^3+16a^2p^6 \\
6a^2-8ap^3-5t & \qquad -30a^2t+40ap^3t+25t^2 \\
 & \qquad -30a^2t+40ap^3t+25t^2
\end{array}$$

20.

$$4y^4x^2-12y^3x^3+17y^2x^4-12yx^5+4x^6\,(2y^2x-3yx^2+2x^3$$

$$\begin{array}{ll}
 & 4y^4x^2 \\
4y^2x-3yx^2 & -12y^3x^3+17y^2x^4 \\
 & -12y^3x^3+9y^2x^4 \\
4y^2x-6yx^2+2x^3 & \qquad 8y^2x^4-12yx^5+4x^6 \\
 & \qquad 8y^2x^4-12yx^5+4x^6
\end{array}$$

21.

$$
\begin{array}{l|l}
 & 25x^4y^2 - 30x^3y^3 + 29x^2y^4 - 12xy^5 + 4y^6 \,(5x^2y - 3xy^2 + 2y^3 \\
 & 25x^4y^2 \\
\hline
10x^2y - 3xy^2 & \quad -30x^3y^3 + 29x^2y^4 \\
 & \quad -30x^3y^3 + \;\; 9x^2y^2 \\
\hline
10x^2y - 6xy^2 + 2y^3 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\
 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\
\hline
\end{array}
$$

22.

$$
\begin{array}{l|l}
 & 16x^4 - 24x^3y + 25x^2y^2 - 12xy^3 + 4y^4 \,(4x^2 - 3xy + 2y^2 \\
 & 16x^4 \\
\hline
8x^2 - 3xy & \quad -24x^3y + 25x^2y^2 \\
 & \quad -24x^3y + \;\; 9x^2y^2 \\
\hline
8x^2 - 6xy + 2y^2 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\
 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\
\hline
\end{array}
$$

23.

$$
\begin{array}{l|l}
 & 9a^2 - 12ab + 24ac - 16bc + 4b^2 + 16c^2 \,(3a - 2b + 4c \\
 & 9a^2 \\
\hline
6a - 2b & \quad -12ab + 4b^2 \\
 & \quad -12ab + 4b^2 \\
\hline
6a - 4b + 4c & \qquad\qquad 24ac - 16bc + 16c^2 \\
 & \qquad\qquad 24ac - 16bc + 16c^2 \\
\hline
\end{array}
$$

24.

$$
\begin{array}{l|l}
 & x^4 + 19x^2 + 25 - 6x^3 - 30x \,(x^2 - 3x + 5 \\
 & x^4 \\
\hline
2x^2 - 3x & \quad -6x^3 + 19x^2 \\
 & \quad -6x^3 + \;\; 9x^2 \\
\hline
2x^2 - 6x + 5 & \qquad\qquad 10x^2 - 30x + 25 \\
 & \qquad\qquad 10x^2 - 30x + 25 \\
\hline
\end{array}
$$

25.

$$25x^2 - 20xy + 4y^2 + 9z^2 - 12yz + 30xz\ (5x - 2y + 3z$$
$$25x^2$$

$$\begin{array}{l|l} 10x - 2y & -20xy + 4y^2 \\ & \underline{-20xy + 4y^2} \\ 10x - 4y + 3z & \quad 30xz - 12yz + 9z^2 \\ & \quad \underline{30xz - 12yz + 9z^2} \end{array}$$

26.

$$4x^4 - 4x^2y + 4x^2y^2 + y^4 - 2y^3 + y^2\ (2x^2 - y + y^2$$
$$4x^4$$

$$\begin{array}{l|l} 4x^2 - y & -4x^2y + y^2 \\ & \underline{-4x^2y + y^2} \\ 4x^2 - 2y + y^2 & \quad 4x^2y^2 - 2y^3 + y^4 \\ & \quad \underline{4x^2y^2 - 2y^3 + y^4} \end{array}$$

LXXXI.

1.

$$4a^6 + \frac{a^2b^4}{16} - a^4b^2\ \left(2a^3 - \frac{ab^2}{4}\right.$$
$$4a^6$$

$$\begin{array}{l|l} 4a^3 - \dfrac{ab^2}{4} & -a^4b^2 + \dfrac{a^2b^4}{16} \\ & \underline{-a^4b^2 + \dfrac{a^2b^4}{16}} \end{array}$$

2.

$$\frac{9}{a^2} - 2 + \frac{a^2}{9}\left(\frac{3}{a} - \frac{a}{3}\right.$$
$$\frac{9}{a^2}$$

$$\begin{array}{l|l} \dfrac{6}{a} - \dfrac{a}{3} & -2 + \dfrac{a^2}{9} \\ & \underline{-2 + \dfrac{a^2}{9}} \end{array}$$

3. $a^4 - 2 + \frac{1}{a^4}\left(a^2 - \frac{1}{a^2}\right.$

$$a^4$$

$$2a^2 - \frac{1}{a^2} \quad \Big| \quad -2 + \frac{1}{a^4}$$

$$-2 + \frac{1}{a^4}$$

4. $\frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}\left(\frac{a}{b} + \frac{b}{a}\right.$

$$\frac{a^2}{b^2}$$

$$\frac{2a}{b} + \frac{b}{a} \quad \Big| \quad 2 + \frac{b^2}{a^2}$$

$$2 + \frac{b^2}{a^2}$$

5. $x^4 - 2x^3 + 2x^2 - x + \frac{1}{4}\ \left(x^2 - x + \frac{1}{2}\right.$

$$x^4$$

$$2x^2 - x \quad \Big| \quad -2x^3 + 2x^2$$

$$-2x^3 + x^2$$

$$2x^2 - 2x + \frac{1}{2} \quad \Big| \quad x^2 - x + \frac{1}{4}$$

$$x^2 - x + \frac{1}{4}$$

6. $x^4 + 2x^3 - x + \frac{1}{4}\ \left(x^2 + x - \frac{1}{2}\right.$

$$x^4$$

$$2x^2 + x \quad \Big| \quad 2x^3 - x$$

$$2x^3 + x^2$$

$$2x^2 + 2x - \frac{1}{2} \quad \Big| \quad -x^2 - x + \frac{1}{4}$$

$$-x^2 - x + \frac{1}{4}$$

7.

$$4a^2 - 12ab + ab^2 + 9b^2 - \frac{3b^3}{2} + \frac{b^4}{16}\left(2a - 3b + \frac{b^2}{4}\right.$$

$$\begin{array}{l|l}
 & 4a^2 \\
\hline
4a - 3b & -12ab + ab^2 + 9b^2 \\
 & -12ab + 9b^2 \\
\hline
4a - 6b + \frac{b^2}{4} & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
 & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
\hline
\end{array}$$

8.

$$x^4 + 8x^2 + 24 + \frac{16}{x^4} + \frac{32}{x^2}\left(x^2 + 4 + \frac{4}{x^2}\right.$$

$$\begin{array}{l|l}
 & x^4 \\
\hline
2x^2 + 4 & 8x^2 + 24 \\
 & 8x^2 + 16 \\
\hline
2x^2 + 8 + \frac{4}{x^2} & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
 & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
\hline
\end{array}$$

9.

$$\frac{16}{9}a^6x^2 + \frac{16}{3}a^5x + 4a^4 - 2a^3x - 3a^2 + \frac{9}{16}\left(\frac{4a^3x}{3} + 2a^2 - \frac{3}{4}\right.$$

$$\begin{array}{l|l}
 & \frac{16}{9}a^6x^2 \\
\hline
\frac{8a^3x}{3} + 2a^2 & \frac{16}{3}a^5x + 4a^4 \\
 & \frac{16}{3}a^5x + 4a^4 \\
\hline
\frac{8a^3x}{3} + 4a^2 - \frac{3}{4} & -2a^3x - 3a^2 + \frac{9}{16} \\
 & -2a^3x - 3a^2 + \frac{9}{16} \\
\hline
\end{array}$$

10.

$$\frac{1}{x^2}-\frac{4}{xy}+\frac{4}{y^2}+\frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}\Big(\frac{1}{x}-\frac{2}{y}+\frac{3}{z}$$

$$\frac{1}{x^2}$$

$$\frac{2}{x}-\frac{2}{y} \quad \Big\backslash \quad -\frac{4}{xy}+\frac{4}{y^2}$$

$$-\frac{4}{xy}+\frac{4}{y^2}$$

$$\frac{2}{x}-\frac{4}{y}+\frac{3}{z} \quad \Big\backslash \quad \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}$$

$$\frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}$$

11.

$$36m^2-\frac{48m}{n}+\frac{16}{n^2}+\frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}\Big(6m-\frac{4}{n}+\frac{p}{5}$$

$$36m^2$$

$$12m-\frac{4}{n} \quad \Big\backslash \quad -\frac{48m}{n}+\frac{16}{n^2}$$

$$-\frac{48m}{n}+\frac{16}{n^2}$$

$$12m-\frac{8}{n}+\frac{p}{5} \quad \Big\backslash \quad \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}$$

$$\frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}$$

12.

$$a^2b^2-6abcd+9c^2d^2+\frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}\,\Big(ab-3cd+\frac{ef}{7}$$

$$a^2b^2$$

$$2ab-3cd \quad \Big\backslash \quad -6abcd+9c^2d^2$$

$$-6abcd+9c^2d^2$$

$$2ab-6cd+\frac{ef}{7} \quad \Big\backslash \quad \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}$$

$$\frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}$$

13.

$$\frac{4x^2}{z^2}-\frac{12xy}{z^2}+\frac{9y^2}{z^2}+4-\frac{6y}{x}+\frac{z^2}{x^2}\left(\frac{2x}{z}-\frac{3y}{z}+\frac{z}{x}\right.$$

$$\frac{4x^2}{z^2}$$

$$\begin{array}{l|l}
\frac{4x}{z}-\frac{3y}{z} & -\frac{12xy}{z^2}+\frac{9y^2}{z^2} \\
 & -\frac{12xy}{z^2}+\frac{9y^2}{z^2} \\
\hline
\frac{4x}{z}-\frac{3y}{z}+\frac{z}{x} & 4-\frac{6y}{x}+\frac{z^2}{x^2} \\
 & 4-\frac{6y}{x}+\frac{z^2}{x^2} \\
\hline
\end{array}$$

14.

$$\frac{4m^2}{n^2}-\frac{16m}{n}+4+\frac{24n}{m}+\frac{9n^2}{m^2}\left(\frac{2m}{n}-4-\frac{3n}{m}\right.$$

$$\frac{4m^2}{n^2}$$

$$\begin{array}{l|l}
\frac{4m}{n}-4 & -\frac{16m}{n}+4 \\
 & -\frac{16m}{n}+16 \\
\hline
\frac{4m}{n}-8-\frac{3n}{m} & -12+\frac{24n}{m}+\frac{9n^2}{m^2} \\
 & -12+\frac{24n}{m}+\frac{9n^2}{m^2} \\
\hline
\end{array}$$

$$\frac{a^2}{9}-\frac{ab}{6}+\frac{b^2}{16}+\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}\left(\frac{a}{3}-\frac{b}{4}+\frac{c}{5}-\frac{d}{2}\right.$$

$$\frac{a^2}{9}$$

$$\frac{2a}{3}-\frac{b}{4} \quad \left|\quad -\frac{ab}{6}+\frac{b^2}{16}\right.$$

$$-\frac{ab}{6}+\frac{b^2}{16}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{c}{5} \quad \left|\quad \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}\right.$$

$$\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{2c}{5}-\frac{d}{2} \quad \left|\quad -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}\right.$$

$$-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

$$49x^4-28x^3-17x^2+6x+\frac{9}{4}\left(7x^2-2x-\frac{3}{2}\right.$$

$$49x^4$$

$$14x^2-2x \quad \left|\quad -28x^3-17x^2\right.$$

$$-28x^3+4x^2$$

$$14x^2-4x-\frac{3}{2} \quad \left|\quad -21x^2+6x+\frac{9}{4}\right.$$

$$-21x^2+6x+\frac{9}{4}$$

$$9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2\left(3x^2-\frac{ax}{2}+bx\right.$$

$$9x^4$$

$$6x^2-\frac{ax}{2} \quad \left|\quad -3ax^3+\frac{a^2x^2}{4}\right.$$

$$-3ax^3+\frac{a^2x^2}{4}$$

$$6x^2-ax+bx \quad \left|\quad 6bx^3-abx^2+b^2x^2\right.$$

$$6bx^3-abx^2+b^2x^2$$

18.

$$9x^4 - 2x^3 - \frac{161}{9}x^2 + 2x + 9 \left(3x^2 - \frac{x}{3} - 3\right.$$

$$\begin{array}{l|l}
 & 9x^4 \\
\hline
6x^2 - \frac{x}{3} & -2x^3 - \frac{161}{9}x^2 \\
 & -2x^3 + \frac{x^2}{9} \\
\hline
6x^2 - \frac{2x}{3} - 3 & -18x^2 + 2x + 9 \\
 & -18x^2 + 2x + 9 \\
\hline
\end{array}$$

LXXXIII.

1.

$$a^3 - 3a^2b + 3ab^2 - b^3 \left(a - b\right.$$

$$\begin{array}{l|l}
 & a^3 \\
\hline
3a^2 - 3ab + b^2 & -3a^2b + 3ab^2 - b^3 \\
 & -3a^2b + 3ab^2 - b^3 \\
\hline
\end{array}$$

2.

$$8a^3 + 12a^2 + 6a + 1 \left(2a +\right.$$

$$\begin{array}{l|l}
 & 8a^3 \\
\hline
12a^2 + 6a + 1 & 12a^2 + 6a + 1 \\
 & 12a^2 + 6a + 1 \\
\hline
\end{array}$$

3.

$$a^3 + 24a^2b + 192ab^2 + 512b^3 \left(a + 8b\right.$$

$$\begin{array}{l|l}
 & a^3 \\
\hline
3a^2 + 24ab + 64b^2 & 24a^2b + 192ab^2 + 512b^3 \\
 & 24a^2b + 192ab^2 + 512b^3 \\
\hline
\end{array}$$

4.

$$a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \left(a + b +\right.$$

$$\begin{array}{l|l}
 & a^3 \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\
\hline
3a^2 + 6ab + 3b^2 & 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \\
\quad + 3ac + 3bc + c^2 & 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \\
\hline
\end{array}$$

5.

$$x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 \left(x - y\right.$$

$$\begin{array}{l|l}
 & x^3 \\
\hline
3x^2 - 3xy + y^2 & -3x^2y + 3xy^2 - y^3 \\
 & -3x^2y + 3xy^2 - y^3 \\
\hline
3x^2 - 6xy + 3y^2 & 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 \\
\quad + 3xz - 3yz + z^2 & 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 \\
\hline
\end{array}$$

$27x^6-54x^5+63x^4-44x^3+21x^2-6x+1\,(3x^2-2x+1$

$27x^6$

$8x^3+4x^2$ $\quad -54x^5+63x^4-44x^3$

$-54x^5+36x^4-8x^3$

$6x^3+21x^2-6x+1$ $\quad 27x^4-36x^3+21x^2-6x+1$

$27x^4-36x^3+21x^2-6x+1$

$1-3a+6a^2-7a^3+6a^4-3a^5+a^6\,(1-a+a^2$

1

$-a^2$ $\quad -3a+6a^2-7a^3$

$-3a+3a^2-a^3$

$-6a^2-3a^3+a^4$ $\quad 3a^2-6a^3+6a^4-3a^5+a^6$

$3a^2-6a^3+6a^4-3a^5+a^6$

$x^3-3x^2y+3xy^2-y^3+8z^3+6x^2z-12xyz+6y^2z+12xz^2-12yz^2$

x^3 $\quad (x-y+2z$

$xy+y^2$ $\quad -3x^2y+3xy^2-y^3$

$-3x^2y+3xy^2-y^3$

$xy+3y^2$ $\quad 6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3$

$z-6yz+4z^2$ $\quad 6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3$

$a^6-12a^5+54a^4-112a^3+108a^2-48a+8\,(a^2-4a+2$

a^6

$2a^3+16a^2$ $\quad -12a^5+54a^4-112a^3$

$-12a^5+48a^4-64a^3$

$4a^3+54a^2-24a+8$ $\quad 6a^4-48a^3+108a^2-48a+8$

$6a^4-48a^3+108a^2-48a+8$

10.

$$8m^6 - 36m^5 + 66m^4 - 63m^3 + 33m^2 - 9m + 1 \,(2m^2 - 3m + 1$$
$$8m^6$$

$12m^4 - 18m^3 + 9m^2$

$$-36m^5 + 66m^4 - 63m^3$$
$$-36m^5 + 54m^4 - 27m^3$$

$12m^4 - 36m^3 + 33m^2 - 9m + 1$

$$12m^4 - 36m^3 + 33m^2 - 9m + 1$$
$$12m^4 - 36m^3 + 33m^2 - 9m + 1$$

11.

$$x^3 + 6x^2y + 12xy^2 + 8y^3 - 3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$$
$$x^3 \qquad (x + 2y - z$$

$3x^2 + 6xy + 4y^2$

$$6x^2y + 12xy^2 + 8y^3$$
$$6x^2y + 12xy^2 + 8y^3$$

$3x^2 + 12xy + 12y^2 - 3xz - 6yz + z^2$

$$-3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$$
$$-3x^2z - 12xyz - 12y^2z + 3xz^2 + 6yz^2 - z^3$$

12.

$$8m^3 - 36m^2n + 54mn^2 - 27n^3 - 12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3 \,(2m - 3n - r$$
$$8m^3$$

$12m^2 - 18mn + 9n^2$

$$-36m^2n + 54mn^2 - 27n^3$$
$$-36m^2n + 54mn^2 - 27n^3$$

$12m^2 - 36mn + 27n^2 - 6mr - 9nr + r^2$

$$-12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$$
$$-12m^2r + 36mnr - 27n^2r + 6mr^2 - 9nr^2 - r^3$$

13.

$$m^3 + 3m^2 - 5 + \frac{3}{m^2} - \frac{1}{m^3} \left(m + 1 - \frac{1}{m}\right.$$
$$m^3$$

$3m^2 + 3m + 1$

$$3m^2 - 5 + \frac{3}{m^2}$$
$$3m^2 + 3m + 1$$

$3m^2 + 6m - \frac{3}{m} + \frac{1}{m^2}$

$$-3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$$
$$-3m - 6 + \frac{3}{m^2} - \frac{1}{m^3}$$

LXXXIV.

1

$$
\begin{array}{lll}
 & 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 \,(4a^2 - 12ax + 9x^2 & \\
 & 16a^4 & \\
\hline
8a^2 - 12ax & \quad -96a^3x + 216a^2x^2 & \\
 & \quad -96a^3x + 144a^2x^2 & \\
\hline
8a^2 - 24ax + 9x^2 & \qquad\qquad 72a^2x^2 - 216ax^3 + 81x^4 & \\
 & \qquad\qquad 72a^2x^2 - 216ax^3 + 81x^4 & \\
\hline
\end{array}
$$

and the square root of $4a^2 - 12ax + 9x^2$ is $2a - 3x$.

2.

$$
\begin{array}{ll}
 & 1 - 8a + 24a^2 - 32a^3 + 16a^4 \,(1 - 4a + 4a^2 \\
 & 1 \\
\hline
2 - 4a & \quad -8a + 24a^2 \\
 & \quad -8a + 16a^2 \\
\hline
2 - 8a + 4a^2 & \qquad\qquad 8a^2 - 32a^3 + 16a^4 \\
 & \qquad\qquad 8a^2 - 32a^3 + 16a^4 \\
\hline
\end{array}
$$

and the square root of $1 - 4a + 4a^2$ is $1 - 2a$.

3.

$$
\begin{array}{ll}
 & 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4 \,(25 + 40x + 16x^2 \\
 & 625 \\
\hline
50 + 40x & \quad 2000x + 2400x^2 \\
 & \quad 2000x + 1600x^2 \\
\hline
50 + 80x + 16x^2 & \qquad\qquad 800x^2 + 1280x^3 + 256x^4 \\
 & \qquad\qquad 800x^2 + 1280x^3 + 256x^4 \\
\hline
\end{array}
$$

and the square root of $25 + 40x + 16x^2$ is $5 + 4x$.

4. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 (a^3 - 3a^2b + 3ab^2 - b^3$

a^6

$2a^3 - 3a^2b$ | $-6a^5b + 15a^4b^2$

$-6a^5b + 9a^4b^2$

$2a^3 - 6a^2b + 3ab^2$ | $6a^4b^2 - 20a^3b^3 + 15a^2b^4$

$6a^4b^2 - 18a^3b^3 + 9a^2b^4$

$2a^3 - 6a^2b + 6ab^2 - b^3$ | $-2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6$

$-2a^3b^3 + 6a^2b^4 - 6ab^5 + b^6$

and the cube root of $a^3 - 3a^2b + 3ab^2 - b^3$ is $a - b$.

5. $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 (x^3 + 3x^2 + 3x + 1$

x^6

$2x^3 + 3x^2$ | $6x^5 + 15x^4$

$6x^5 + 9x^4$

$2x^3 + 6x^2 + 3x$ | $6x^4 + 20x^3 + 15x^2$

$6x^4 + 18x^3 + 9x^2$

$2x^3 + 6x^2 + 6x + 1$ | $2x^3 + 6x^2 + 6x + 1$

$2x^3 + 6x^2 + 6x + 1$

and the cube root of $x^3 + 3x^2 + 3x + 1$ is $x + 1$.

6. $m^6 - 12m^5 + 60m^4 - 160m^3 + 240m^2 - 192m + 64 (m^3 - 6m^2 + 12m - 8$

m^6

$2m^3 - 6m^2$ | $-12m^5 + 60m^4$

$-12m^5 + 36m^4$

$2m^3 - 12m^2 + 12m$ | $24m^4 - 160m^3 + 240m^2$

$24m^4 - 144m^3 + 144m^2$

$2m^3 - 12m^2 + 24m - 8$ | $-16m^3 + 96m^2 - 192m + 64$

$-16m^3 + 96m^2 - 192m + 64$

and the cube root of $m^3 - 6m^2 + 12m - 8$ is $m - 2$.

LXXXV.

1. $x = \pm 8$.
2. $x = \pm ab$.
3. $x^2 = 10000$; $x = \pm 100$.
4. $x^2 = 49$; $x = \pm 7$.
5. $3x^2 = 33$; $x^2 = 11$; $x = \pm\sqrt{11}$.
6. $x^2 = 64a^4c^6$, etc.
7. $4x^2 - 48 = 3x^2 - 12$; $x^2 = 36$, etc.
8. $250000 - x^2 = 233359$; $x^2 = 16641$; $x = \pm 129$.
9. $8112 = 3x^2$; $x^2 = 2704$, etc.
10. $\frac{11x^2}{2} - 18x + 65 = 9x^2 - 18x + 9$; $11x^2 - 18x^2 = -112$; $7x^2 = 112$, etc.
11. $mx^2 = q - n$; $x^2 = \frac{q-n}{m}$, etc.
12. $x^2 - ax + b = ax^2 - ax$; $ax^2 - x^2 = b$; $x^2 = \frac{b}{a-1}$, etc.
13. $180x^2 - 225 = 114x^2 + 171$; $66x^2 = 396$; $x^2 = 6$, etc.
14. $42x^2 - 126 = 35x^2 - 70$; $7x^2 = 56$; $x^2 = 8$; $x^2 = 4 \times 2$; $x = \pm 2\sqrt{2}$.

LXXXVI.

1. $x^2 + 6x + 9 = 81$; $x + 3 = \pm 9$, etc.
2. $x^2 + 12x + 36 = 100$; $x + 6 = \pm 10$, etc.
3. $x^2 + 14x + 49 = 64$; $x + 7 = \pm 8$, etc.
4. $x^2 + 46x + 529 = 625$; $x + 23 = \pm 25$, etc.
5. $x^2 + 128x + 4096 = 4489$; $x + 64 = \pm 67$, etc.
6. $x^2 + 8x + 16 = 81$; $x + 4 = \pm 9$, etc.
7. $x^2 + 18x + 81 = 324$; $x + 9 = \pm 18$, etc.
8. $x^2 + 16x + 64 = 484$; $x + 8 = \pm 22$, etc.

LXXXVII.

1. $x^2-6x+9=16$; $x-3=\pm 4$, etc.

2. $x^2-4x+4=9$; $x-2=\pm 3$, etc.

3. $x^2-20x+100=121$; $x-10=\pm 11$, etc.

4. $x^2-2x+1=64$; $x-1=\pm 8$, etc.

5. $x^2-12x+36=4$; $x-6=\pm 2$, etc.

6. $x^2-14x+49=4$; $x-7=\pm 2$, etc.

7. $x^2-234x+13689=1$; $x-117=\pm 1$, etc.

8. $x^2-5x+6=15x+42$; $x^2-20x=36$; $x^2-20x+100=136$;
$x-10=\pm\sqrt{4\times 34}$; $x=10\pm 2\sqrt{34}$.

9. $3x^2-17x-2x^2-5x+120=0$; $x^2-22x+121=1$, etc.

10. $x^2-10x+25+x^2-14x+49=x^2-8x+46$; $x^2-16x=-28$, etc.

LXXXVIII.

1. $x^2+7x+\frac{49}{4}=\frac{169}{4}$; $x+\frac{7}{2}=\pm\frac{13}{2}$, etc.

2. $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

3. $x^2+9x+\frac{81}{4}=\frac{256}{4}$; $x+\frac{9}{2}=\pm\frac{16}{2}$, etc.

4. $x^2-13x+\frac{169}{4}=\frac{729}{4}$; $x-\frac{13}{2}=\pm\frac{27}{2}$, etc.

5. $x^2+x+\frac{1}{4}=\frac{9}{16}$; $x+\frac{1}{2}=\pm\frac{3}{4}$, etc.

$x^2-x+\frac{1}{4}=\frac{289}{4}$; $x-\frac{1}{2}=\pm\frac{17}{2}$, etc.

$x^2+37x+\frac{1369}{4}=\frac{16129}{4}$; $x+\frac{37}{2}=\pm\frac{127}{2}$, etc.

$x^2-x+\frac{1}{4}=\frac{225}{4}$; $x-\frac{1}{2}=\pm\frac{15}{2}$, etc.

$5x-x^2+2x^2-14x-10x+60=0$; $x^2-19x=-60$;

$x^2-19x+\frac{361}{4}=\frac{121}{4}$, etc.

$35x^2-312x+693-34x^2+21x+45=448$; $x^2-291x=-290$;

$x^2-291x+\frac{84681}{4}=\frac{83521}{4}$; $x-\frac{291}{2}=\pm\frac{289}{2}$, etc.

LXXXIX.

1. $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{36}{9}$; $x-\frac{1}{3}=\pm\frac{6}{3}$, etc.

2. $x^2+\frac{4}{5}x+\frac{4}{25}=\frac{1}{25}$; $x+\frac{2}{5}=\pm\frac{1}{5}$, etc.

3. $x^2-\frac{28x}{9}+\frac{196}{81}=\frac{169}{81}$; $x-\frac{14}{9}=\pm\frac{13}{9}$, etc.

4. $x^2-\frac{8}{11}x+\frac{16}{121}=\frac{49}{121}$; $x-\frac{4}{11}=\pm\frac{7}{11}$, etc.

5. $x^2+\frac{4}{35}x+\frac{4}{1225}=\frac{529}{1225}$; $x+\frac{2}{35}=\pm\frac{23}{35}$, etc.

6. $x^2-\frac{16}{5}x+\frac{64}{25}=\frac{144}{25}$; $x-\frac{8}{5}=\pm\frac{12}{5}$, etc.

7. $x^2-\frac{26}{3}x+\frac{169}{9}=\frac{121}{9}$; $x-\frac{13}{3}=\pm\frac{11}{3}$, etc.

8. $x^2-\frac{4}{7}x+\frac{4}{49}=\frac{2209}{49}$; $x-\frac{2}{7}=\pm\frac{47}{7}$, etc.

XC.

1. $x^2-\frac{1}{3}x+\frac{1}{36}=\frac{289}{36}$; $x-\frac{1}{6}=\pm\frac{17}{6}$, etc.

2. $x^2-\frac{1}{5}x+\frac{1}{100}=\frac{9801}{100}$; $x-\frac{1}{10}=\pm\frac{99}{10}$, etc.

3. $x^2+\frac{1}{2x}\ +\frac{1}{16}=\frac{625}{16}$; $x+\frac{1}{4}=\pm\frac{25}{4}$, etc.

4. $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{1225}{16}$; $x+\frac{3}{4}=\pm\frac{35}{4}$, etc.

5. $x^2-\frac{9}{5}x+\frac{81}{100}=\frac{1681}{100}$; $x-\frac{9}{10}=\pm\frac{41}{10}$, etc.

6. $x^2-\frac{11}{2}x+\frac{121}{16}=\frac{25}{16}$; $x-\frac{11}{4}=\pm\frac{5}{4}$, etc.

7. $x^2-\frac{15}{4}x+\frac{225}{64}=\frac{2401}{64}$; $x-\frac{15}{8}=\pm\frac{49}{8}$, etc.

8. $x^2-\frac{23}{7}x+\frac{529}{196}=\frac{676}{196}$; $x-\frac{23}{14}=\pm\frac{26}{14}$, etc.

XCI.

1. $x^2+2ax+a^2=2a^2$; $x+a=\pm\sqrt{2}.a$, etc.

2. $x^2-4ax+4a^2=11a^2$; $x-2a=\pm\sqrt{11}.a$, etc.

3. $x^2+3mx+\frac{9m^2}{4}=4m^2$; $x+\frac{3m}{2}=\pm 2m$, etc.

4. $x^2-\frac{5nx}{2}+\frac{25n^2}{16}=\frac{49n^2}{16}$; $x-\frac{5n}{4}=\pm\frac{7n}{4}$, etc.

$$x^2+(a-1)x+\frac{(a-1)^2}{4}=\frac{a^2+2a+1}{4}\ ;\ x+\frac{a-1}{2}=\pm\frac{a+1}{2},\text{ etc.}$$

$$x^2+(a-b)x+\frac{(a-b)^2}{4}=\frac{(a+b)^2}{4}\ ;\ x+\frac{a-b}{2}=\pm\frac{a+b}{2},\text{ etc.}$$

$\dfrac{a^2}{(x+a)^2}=\dfrac{b^2}{(x-a)^2}$; and, taking the square root of each side,

$$\frac{a}{x+a}=\pm\frac{b}{x-a},\text{ etc.}$$

$$acx^2-adx+bcx=bd\ ;\ x^2+\frac{bc-ad}{ac}x=\frac{bd}{ac}\ ;$$

$$x^2+\frac{bc-ad}{ac}x+\frac{(bc-ad)^2}{4a^2c^2}=\frac{(bc+ad)^2}{4a^2c^2}\ ;$$

$$x+\frac{bc-ad}{2ac}=\pm\frac{bc+ad}{2ac}\ ;\ \ x=\frac{2ad}{ac}\text{ or }-\frac{2bc}{2ac},\text{ etc.}$$

$$(a+b)x^2-cx=\frac{ac}{a+b}\ ;\ \ x^2-\frac{cx}{a+b}=\frac{ac}{(a+b)^2}\ ;$$

$$x^2-\frac{cx}{a+b}+\frac{c^2}{4(a+b)^2}=\frac{c^2+4ac}{4(a+b)^2}\ ;\ x-\frac{c}{2(a+b)}=\pm\frac{\sqrt{c^2+4ac}}{2(a+b)},\text{ etc.}$$

$$x^2-\frac{2b^2x}{ac}=-\frac{b^4}{a^2c^2}\ ;\ \ x^2-\frac{2b^2x}{ac}+\frac{b^4}{a^2c^2}=0\ ;\ x-\frac{b^2}{ac}=0\ ;\ x=\frac{b^2}{ac}.$$

$$x^2+\frac{3a^2+b^2}{abc}x=\frac{6a^2+ab-2b^2}{abc^2}\ ;$$

$$x^2+\frac{3a^2+b^2}{abc}x+\frac{(3a^2+b^2)^2}{4a^2b^2c^2}=\frac{24a^3b+4a^2b^2-8ab^3+9a^4+6a^2b^2+b^4}{4a^2b^2c^2}$$

$$x+\frac{3a^2+b^2}{2abc}=\pm\frac{3a^2+4ab-b^2}{2abc},\text{ etc.}$$

12. $x^2+\dfrac{4a^2c^2+4abd^2}{4a^2-9cd^2}x=-\dfrac{a^2c^4+2abc^2d^2+b^2d^4}{4a^2-9cd^2}$;

$$x^2+\frac{4a^2c^2+4abd^2}{4a^2-9cd^2}x+\frac{(2a^2c^2+2abd^2)^2}{(4a^2-9cd^2)^2}=\frac{9cd^2(a^2c^4+2abc^2d^2+b^2d^4)}{(4a^2-9cd^2)^2};$$

$$x+\frac{2a^2c^2+2abd^2}{4a^2-9cd^2}=\pm\frac{3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{-2a(ac^2+bd^2)\pm 3d\sqrt{c}.(ac^2+bd^2)}{4a^2-9cd^2};$$

$$x=\frac{(-2a\pm 3d\sqrt{c})(ac^2+bd^2)}{(2a+3d\sqrt{c})(2a-3d\sqrt{c})},\text{ etc.}$$

XCII.

1. $x^2-7x=8$; $x^2-7x+\frac{49}{4}=\frac{81}{4}$; $x-\frac{7}{2}=\pm\frac{9}{2}$, etc.

2. $x^2-5x=6$; $x^2-5x+\frac{25}{4}=\frac{49}{4}$; $x-\frac{5}{2}=\pm\frac{7}{2}$, etc.

3. $x^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

4. $x^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

5. $x^2+7x=18$; $x^2+7x+\frac{49}{4}=\frac{121}{4}$; $x+\frac{7}{2}=\pm\frac{11}{2}$, etc.

6. $4x^2-12x-12+x=22x-66$; $4x^2-33x=-54$; $x^2-\frac{33x}{4}=-\frac{54}{4}$

$$x^2-\frac{33x}{4}+\frac{1089}{64}=\frac{225}{64};\ x-\frac{33}{8}=\pm\frac{15}{8},\text{ etc.}$$

7. $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

8. $10x^2-30x-6x+6=7x^2-27x+18$; $3x^2-9x=12$; $x^2-3x=4$, e

$^2-6x=16$; $x^2-6x+9=25$; $x-3=\pm 5$, etc.

$x^2+20x-x^2+9=4x^2+22x+30$; $3x^2-2x=21$;

$x^2-\frac{2x}{3}=7$; $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{64}{9}$; $x-\frac{1}{3}=\pm\frac{8}{3}$, etc.

$x^2+12x-x^2+49=4x^2+34x+42$; $3x^2-22x=-7$;

$x^2-\frac{22x}{3}=-\frac{7}{3}$; $x^2-\frac{22x}{3}+\frac{121}{9}=\frac{100}{9}$; $x-\frac{11}{3}=\pm\frac{10}{3}$, etc.

$^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

$^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

$2x^2-8x+177=192$; $x^2-\frac{2x}{3}=\frac{5}{4}$; $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{49}{36}$, etc.

$x^2-169=26x$; $x^2-\frac{26x}{3}=\frac{169}{3}$; $x^2-\frac{26x}{3}+\frac{169}{9}=\frac{676}{9}$, etc.

$^2-9x=-20$; $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

$x^2-28x-48-300+20x=7x^2-56x+84$; $x^2-48x=-432$, etc.

$^2-8x=-12$; $x^2-8x+16=4$; $x-4=\pm 2$, etc.

$x^2-90x+125-54x^2=9x^2-75x$; $54x^2+15x=125$;

$x^2+\frac{15x}{54}=\frac{125}{54}$; $x^2+\frac{15x}{54}+\frac{225}{11664}=\frac{27225}{11664}$; $x+\frac{15}{108}=\pm\frac{165}{108}$, etc.

$x^2+35x-8x^2+20x=6x^2+16x-70$; $7x^2-39x=70$;

$x^2-\frac{39x}{7}=10$; $x^2-\frac{39x}{7}+\frac{1521}{196}=\frac{3481}{196}$; $x-\frac{39}{14}=\pm\frac{59}{14}$, etc.

$x^2-20x+6x^2+16x-70=7x^2+35x$; $7x^2-39x=70$, etc.

$x^2-6x+9+4x=44$; $x^2-2x=35$, etc.

23. $x^2+11x=7x^2-9-4x$; $6x^2-15x=9$; $2x^2-5x=3$; $x^2-\frac{5x}{2}=\frac{3}{2}$;

$x^2-\frac{5x}{2}+\frac{25}{16}=\frac{49}{16}$; $x-\frac{5}{4}=\pm\frac{7}{4}$, etc.

24. $x^2+\frac{x}{6}=\frac{2}{6}$; $x^2+\frac{x}{6}+\frac{1}{144}=\frac{49}{144}$; $x+\frac{1}{12}=\pm\frac{7}{12}$, etc.

25. $x^2-\frac{x}{2}+\frac{1}{16}=\frac{25}{144}$; $x-\frac{1}{4}=\pm\frac{5}{12}$, etc.

26. $x^2-x+\frac{1}{4}=\frac{841}{4}$; $x-\frac{1}{2}=\pm\frac{29}{2}$, etc.

27. $6x+2x+2=3x^2+3x$; $3x^2-5x=2$; $x^2-\frac{5x}{3}=\frac{2}{3}$, etc.

28. $4x^2-33=x$; $x^2-\frac{x}{4}=\frac{33}{4}$; $x^2-\frac{x}{4}+\frac{1}{64}=\frac{529}{64}$, etc.

29. $2x^3=3x^3-3x+2x^2-4x+2$; $x^2-\frac{7x}{3}=-\frac{2}{3}$, etc.

30. $x^2-\frac{7x}{15}=\frac{46}{15}$; $x^2-\frac{7x}{15}+\frac{49}{900}=\frac{2809}{900}$; $x-\frac{7}{30}=\pm\frac{53}{30}$, etc.

31. $5x+10-10x+20=3x^2-12$; $x^2+\frac{5x}{3}=14$, etc.

32. $4x^3-100+40x-4x^3=75x-15x^2$; $15x^2-35x=100$;

$x^2-\frac{7x}{3}=\frac{20}{3}$; $x^2-\frac{7x}{3}+\frac{49}{36}=\frac{289}{36}$, etc.

33. $90x-126+18x=22x^2$; $x^2-\frac{54x}{11}=-\frac{63}{11}$; $x^2-\frac{54x}{11}+\frac{729}{121}=\frac{36}{12}$.

34. $3x^2-5x=7x+420$; $x^2-4x=140$, etc.

35. $\frac{48-12x+40-8x}{20-9x+x^2}=\frac{32}{x+2}$; $\frac{22-5x}{20-9x+x^2}=\frac{8}{x+2}$;

$44+12x-5x^2=160-72x+8x^2$; $13x^2-84x=-116$;

$x^2-\frac{84x}{13}=-\frac{116}{13}$; $x^2-\frac{84x}{13}+\frac{1764}{169}=\frac{256}{169}$, etc.

$)x^2+490-140x+10x^2=203x-29x^2$; $49x^2-343x=-490$;
$x^2-7x=-10$, etc.

$^2+(a+b)x+\frac{(a+b)^2}{4}=\frac{(a-b)^2}{4}$; $x+\frac{a+b}{2}=\pm\frac{a-b}{2}$, etc.

$^2-(b-a)x+\frac{(b-a)^2}{4}=\frac{(b+a)^2}{4}$; $x-\frac{b-a}{2}=\pm\frac{b+a}{2}$, etc.

$^2-2ax+a^2=b^2$; $x-a=\pm b$, etc.

$^2-(a^2-a^3)x+\frac{(a^2-a^3)^2}{4}=\frac{(a^2+a^3)^2}{4}$; $x-\frac{a^2-a^3}{2}=\pm\frac{a^2+a^3}{2}$, etc.

$^2+\frac{ax}{b}+\frac{a^2}{4b^2}=\frac{9a^2}{4b^2}$; $x+\frac{a}{2b}=\pm\frac{3a}{2b}$, etc.

$^2-\frac{a^2+b^2}{ab}x+\frac{(a^2+b^2)^2}{4a^2b^2}=\frac{(a^2-b^2)^2}{4a^2b^2}$; $x-\frac{a^2+b^2}{2ab}=\pm\frac{a^2-b^2}{2ab}$.

XCIII.

$\left.\begin{array}{l}+2xy+y^2=1600\\ 4xy \quad =1200\end{array}\right\}$, $x^2-2xy+y^2=400$, $\left\{\begin{array}{l}x+y=\quad 40\\ x-y=\pm 20\end{array}\right.$, etc.

$\left.\begin{array}{l}+2xy+y^2=169\\ 4xy \quad =144\end{array}\right\}$, $x^2-2xy+y^2=25$, $\left\{\begin{array}{l}x+y=\quad 13\\ x-y=\pm\ 5\end{array}\right.$, etc.

$\left.\begin{array}{l}+2xy+y^2=841\\ 4xy \quad =400\end{array}\right\}$, $x^2-2xy+y^2=441$, $\left\{\begin{array}{l}x+y=\quad 29\\ x-y=\pm 21\end{array}\right.$, etc.

$\left.\begin{array}{l}-2xy+y^2=361\\ 4xy \quad =264\end{array}\right\}$, $x^2+2xy+y^2=625$, $\left\{\begin{array}{l}x-y=\quad 19\\ x+y=\pm 25\end{array}\right.$, etc.

$\left.\begin{array}{l}-2xy+y^2=2025\\ 4xy \quad =1000\end{array}\right\}$, $x^2+2xy+y^2=3025$, $\left\{\begin{array}{l}x-y=\quad 45\\ x+y=\pm 55\end{array}\right.$, etc.

$\left.\begin{array}{l}-2xy+y^2=9801\\ 4xy \quad =\ 400\end{array}\right\}$, $x^2+2xy+y^2=10201$, $\left\{\begin{array}{l}x-y=\quad 99\\ x+y=\pm 101\end{array}\right.$, etc.

XCIV.

1. $\left.\begin{array}{l} x^2-2xy+y^2=16 \\ x^2 \qquad\quad +y^2=40 \end{array}\right\}$; $2xy=24$; $x^2+2xy+y^2=64$, etc.

2. $\left.\begin{array}{l} x^2-2xy+y^2=100 \\ x^2 \qquad\quad +y^2=178 \end{array}\right\}$; $2xy=78$; $x^2+2xy+y^2=256$, etc.

3. $\left.\begin{array}{l} x^2-2xy+y^2=196 \\ x^2 \qquad\quad +y^2=436 \end{array}\right\}$; $2xy=240$; $x^2+2xy+y^2=676$, etc.

4. $\left.\begin{array}{l} x^2+2xy+y^2=64 \\ x^2 \qquad\quad +y^2=32 \end{array}\right\}$; $2xy=32$; $x^2-2xy+y^2=0$, etc.

5. $\left.\begin{array}{l} x^2+2xy+y^2=144 \\ x^2 \qquad\quad +y^2=104 \end{array}\right\}$; $2xy=40$; $x^2-2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+2xy+y^2=2401 \\ x^2 \qquad\quad +y^2=1681 \end{array}\right\}$; $2xy=720$; $x^2-2xy+y^2=961$, etc.

XCV.

1. $\left.\begin{array}{l} x^2-\ \ xy+y^2=13 \\ x^2+2xy+y^2=49 \end{array}\right\}$; $3xy=36$; $x^2-2xy+y^2=1$, etc.

2. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 31 \\ x^2+2xy+y^2=121 \end{array}\right\}$; $3xy=90$; $x^2-2xy+y^2=1$, etc.

3. $\left.\begin{array}{l} x^2-\ \ xy+y^2=\ \ 84 \\ x^2+2xy+y^2=144 \end{array}\right\}$; $3xy=60$; $x^2-2xy+y^2=64$, etc.

4. $\left.\begin{array}{l} x^2+\ \ xy+y^2=28 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=24$; $x^2+2xy+y^2=36$, etc.

5. $\left.\begin{array}{l} x^2+\ \ xy+y^2=49 \\ x^2-2xy+y^2=\ \ 4 \end{array}\right\}$; $3xy=45$; $x^2+2xy+y^2=64$, etc.

6. $\left.\begin{array}{l} x^2+\ \ xy+y^2=93 \\ x^2-2xy+y^2=\ \ 9 \end{array}\right\}$; $3xy=84$; $x^2+2xy+y^2=121$, etc.

XCVI.

1. $$\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=\frac{81}{400}\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=\frac{41}{400}\end{aligned}\right\};\ \frac{2}{xy}=\frac{40}{400};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400},\text{ etc.}$$

2. $$\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=\frac{9}{16}\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=\frac{5}{16}\end{aligned}\right\};\ \frac{2}{xy}=\frac{4}{16};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{16},\text{ etc.}$$

3. $$\left.\begin{aligned}\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}&=25\\ \frac{1}{x^2}\qquad+\frac{1}{y^2}&=13\end{aligned}\right\};\ \frac{2}{xy}=12;\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1,\text{ etc.}$$

4. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{12}$, etc.

5. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{2}$, etc.

6. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=7$, etc. +

XCVII.

1. $x-y=1$; $x^2-2xy+y^2=1$; $3xy=36$, etc.

2. $x^2+12xy+36y^2=576$; $x+6y=\pm 24$.

   Now, from the first equation, $x(x+6y)=144$, $\therefore\ \pm 24x=144$, etc.

3. $x^2+2xy+y^2=441$; $x+y=\pm 21$.

   Now, from the first equation, $x(x+y)=210$, $\therefore\ \pm 21x=210$, etc.

4. $x^2+2xy+y^2=100$; and $x^2-2xy+y^2=36$;

   hence $x+y=\pm 10$, and $x-y=\pm 6$, etc.

5. $x+y=8$; $x^2+2xy+y^2=64$; $3xy=45$, etc.

6. $4x=10+5y$; put $\dfrac{10+5y}{4}$ for x in the first equation;

then $\dfrac{(10+5y)^2}{4}+\dfrac{90y+45y^2}{4}=100$; $70y^2+190y=660$;

$7y^2+19y=66$; whence $y=2$ or $-\dfrac{33}{7}$, etc.

7. Put $y=mx$. Then $\dfrac{x^2+mx^2+m^2x^2}{3m^2x^2-5mx^2}=\dfrac{39}{25}$; $\dfrac{1+m+m^2}{3m^2-5m}=\dfrac{39}{25}$;

$25+25m+25m^2=117m^2-195m$; $92m^2-220m=25$;

hence we get $m=\dfrac{5}{2}$ or $-\dfrac{5}{46}$. Taking the value $\dfrac{5}{2}$, we get

$x^2+\dfrac{5x^2}{2}+\dfrac{25x^2}{4}=39$; $39x^2=39\times 4$; $x=\pm 2$, etc.

8. Put $y=mx$. Then $\dfrac{x^2+mx^2}{mx^2-m^2x^2}=\dfrac{66}{5}$; $\dfrac{1+m}{m-m^2}=\dfrac{66}{5}$;

$5+5m=66m-66m^2$; $m^2-\dfrac{61m}{66}=-\dfrac{5}{66}$; hence we get

$m=\dfrac{1}{11}$, or $\dfrac{5}{6}$. Taking the value $\dfrac{5}{6}$, $x^2+\dfrac{5x^2}{6}=66$,

whence $x^2=36$, $x=\pm 6$, etc.

9. Put $y=mx$. Then $\dfrac{3+4m}{5m+2m^2}=\dfrac{20}{12}$; whence $m=\dfrac{1}{2}$ or $-\dfrac{3}{10}$.

Taking the value $\frac{1}{2}$, we get $3x^2+2x^2=20$; $x=\pm 2$, etc.

10. Put $y=mx$. Then $\dfrac{x^2-mx^2+m^2x^2}{3x^2+13mx^2+8m^2x^2}=\dfrac{7}{162}$; $\dfrac{1-m+m^2}{3+13m+8m^2}=\dfrac{7}{162}$

$162-162m+162m^2=21+91m+56m^2$; $106m^2-253m=-141$;

$m^2-\dfrac{253}{136}m=-\dfrac{141}{106}$; $m-\dfrac{253}{212}=\pm\dfrac{65}{212}$; $m=\dfrac{3}{2}$ or $-\dfrac{47}{53}$.

Taking the value $\dfrac{3}{2}$, we get $x^2-\dfrac{3x^2}{2}+\dfrac{9x^2}{4}=7$; $x=\pm 2$, etc.

11. Put $y=mx$. Then $\frac{1-m}{m+m^2}=\frac{35}{18}$; $35m^2+53m=18$;

$m+\frac{53}{70}=\pm\frac{73}{70}$; $m=\frac{2}{7}$ or $-\frac{9}{10}$.

Taking the value $\frac{2}{7}$ we get $x=\pm 7$, etc.

12. $x=\frac{29-7y}{5}$; putting this for x in the first equation, we get

$\frac{2523-1218y+147y^2}{25}+\frac{116y-28y^2}{5}+5y^2=71$;

$132y^2-638y=-748$; $y=2$ or $\frac{187}{66}$, etc.

13. $x+y=22$; $x^2+2xy+y^2=484$; $3xy=360$; $xy=120$;

$x^2-2xy+y^2=4$; $x-y=\pm 2$, etc.

14. Subtract the second equation from the first; $x^2+2xy+y^2=169$; $x+y=\pm 13$. Take the positive value, and for y put $13-x$ in the first equation; $x^2+117x-9x^2=340$; $x^2-\frac{117}{8}x=-\frac{340}{8}$;

$x^2-\frac{117}{8}x+\frac{13689}{256}=\frac{2809}{256}$; $x-\frac{117}{16}=\pm\frac{53}{16}$, etc.

15. $\left.\begin{array}{r}x^2+y^2=225\\ 2xy=216\end{array}\right\}$, $\left.\begin{array}{r}x^2+y^2+2xy=441\\ x^2+y^2-2xy=\ \ 9\end{array}\right\}$, $\left.\begin{array}{r}x+y=\pm 21\\ x-y=\pm\ \ 3\end{array}\right\}$, etc.

XCVIII.

1. Let x be the number; then $\frac{x}{2}\times\frac{x}{3}=864$; $x^2=5184$, etc.

2. Let x be the number; then $\left(\frac{x}{7}\times\frac{x}{8}\right)\div 3=298\frac{2}{3}$; $\frac{x^2}{56}=896$, etc.

3. Let x be the number; then $(94-x)(94+x)=8512$, etc.

4. Let x be the greater; then $\frac{750}{x}$ is the less; and $x \times \frac{x}{750} = 3\frac{1}{3}$, etc.

5. Let x and y be the numbers; then $x^2 + y^2 = 13001$ and $x^2 - y^2 = 1449$; adding, $2x^2 = 14450$, etc.

6. Let x be the greater; then $\frac{377}{x}$ is the less; and $x - 21 = 21 - \frac{377}{x}$; $x^2 - 42x = -377$; $x^2 - 42x + 441 = 64$, etc.

7. Let x be the number; then $\frac{x}{2} \times \frac{x}{3} \times \frac{x}{4} \times \frac{x}{5} = 6750$; $x^4 = 810000$; $x^2 = 900$; $x = 30$.

8. Let x be the number; then $\frac{11500}{x} = x + \frac{51}{x}$; $11500 = x^2 + 51$, etc.

9. Let x be the number; then $(x+20)^2 + 2(x-10)^2 = 17475$; $x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; $3x^2 = 16875$, etc.

10. Let x and $26 - x$ be the numbers; then $x^2 + (26 - x)^2 = 436$; $2x^2 - 52x + 676 = 436$; $x^2 - 26x = -120$, etc.

11. Let $x + 17$ and x be the numbers; then $(x+17)^2 + x^2 = 325$, etc.

12. Let x and $\frac{255}{x}$ be the numbers; then $x^2 + \frac{65025}{x^2} = 514$;
$x^4 - 514x^2 = -65025$; $x^4 - 514x^2 + 66049 = 1024$;
$x^2 - 257 = \pm 32$; $x^2 = 289$ or 225; $x = 17$ or 15.

13. Let x and $16 - x$ be the parts; then $x^2 + (16 - x)^2 + (16 - x)x = 208$
$x^2 - 16x + 256 = 208$; $x^2 - 16x = -48$, etc.

14. Let x^2 be the number; then $x^2 + x = 1332$; $x^2 + x + \frac{1}{4} = \frac{5329}{4}$
$x + \frac{1}{2} = \pm\frac{73}{2}$; $x = 36$; $x^2 = 1296$.

15. Let x^2 be the number; then $x^2 - x = \frac{195}{4}$, etc.

16. Let x^2 be the number; then $x^2 - x = 2550$, etc.

17. Let x and $\frac{24}{x}$ be the numbers; then $\left(x + \frac{24}{x}\right)\left(x - \frac{24}{x}\right) = 20$;

$x^2 - \frac{576}{x^2} = 20$; $x^4 - 20x^2 = 576$; $x^4 - 20x^2 + 100 = 676$;

$x^2 - 10 = \pm 26$; taking the positive value $x = 6$, etc.

18. Let x be the greater, y the less; then $x(x+y) = 204$ and

$y(x-y) = 35$, or, $x^2 + xy = 204$ and $xy - y^2 = 35$.

Put $y = mx$. Then $\frac{x^2 + mx^2}{mx^2 - m^2x^2} = \frac{204}{35}$; hence one value of m is $\frac{5}{12}$

and $x^2 + \frac{5x^2}{12} = 204$; $17x^2 = 2448$; $x^2 = 144$, etc.

19. Let x and $x - 5$ be the numbers; then $x(2x - 5) = 228$, etc.

20. Let $x - 1$, x, $x + 1$ be the numbers; then $(x - 1)x(x + 1) = 3x$;

$x^2 - 1 = 3$; $x^2 = 4$; $x = 2$, etc.

21. Let x and $x + 1$ be the numbers; then $(x + 1)^2 - x^2 = 15$, etc.

22. Let x and $x + 1$ be the numbers; then $x^2 + (x + 1)^2 = 481$, etc.

23. Let $x - 1$, x and $x + 1$ be the numbers; then $(x - 1)^2 + x^2 + (x + 1)^2$

$= 365$, etc.

24. Let x be the number; then $\frac{60}{x} = \frac{60}{x+3} + 1$;

$60x + 180 = 60x + x^2 + 3x$; $x^2 + 3x = 180$, etc.

25. Let x be the number; then $\frac{80}{x} = \frac{80}{x+4} + 1$,

26. Let x be the price of each piece in shillings; then $675 = 48 \times \frac{675}{x} - x$;
$675x = 32400 - x^2$, etc.

27. Let x be the number; then $\frac{180}{x} = \frac{180}{x+3} + 3$, etc.

28. Let x be the number of miles; then $\frac{108}{x} = \frac{108}{x+3} + 6$, etc.

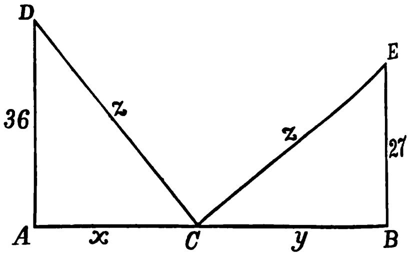
29. Let x be the number of sheep; then $\frac{60}{x} = \frac{54}{x-15} - \frac{1}{10}$, etc.

30. Let x and $x+2$ be the number of hours; then $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{2\frac{11}{12}}$, etc.

31. Let x be the number of yards in the breadth;
then $x\,(x+1) = 10100$, etc.

32. Let $10x + y$ be the number; then $x = 2y$ and $(10x+y)\,(10y+x)$
$= 2268$; hence $21y \times 12y = 2268$; $y^2 = 9$, etc.

33. Let x and y be the distances in feet of the ladder from the bottoms of the houses; z the length of the ladder in feet.



Then since DCE is a right angle,

DCA and ECB together make a right angle. Eucl. i. 13.

But DCA and ADC together make a right angle. Eucl. i. 32.

$\therefore$ angle $ECB =$ angle ADC.

Hence triangles ACD and BEC are equal in all respects. Eucl. i. 26.

$\therefore$ $x = 27$, and $y = 36$; and $x + y = 63$.

Also $z^2 = (27)^2 + (36)^2$. Eucl. i. 47; and $\therefore$ $z = 45$.

34. Let x and y be the length and width in feet.

$$\left.\begin{aligned}\text{Then } \frac{7x}{8} \times \frac{15y}{16} &= xy - \frac{23}{4} \\ \text{and } 2\left(\frac{7x}{8} + \frac{15y}{16}\right) &= 2(x+y) - \frac{17}{4}\end{aligned}\right\};$$

$$\left.\begin{aligned}105xy &= 128xy - 736 \\ 14x + 15y &= 16x + 16y - 34\end{aligned}\right\}; \quad \left.\begin{aligned}xy &= 32 \\ 2x + y &= 34\end{aligned}\right\}; \quad 34x - 2x^2 = 32, \text{ etc.}$$

35. Let $10x+y$ be the number. Since the number is less than 50, and the difference of the digits 4, y is the greater of the digits.

$$\left.\begin{aligned}\text{Then } y - x &= 4 \\ (10y+x)^2 - (10x+y)^2 &= 3960\end{aligned}\right\}, \text{ etc.}$$

36. Let x be the number of rows, $\frac{10000}{x}$ is then the number of trees in each row.

Then $(x-20)\left(\frac{10000}{x} + 25\right) = 10000$

or, $10000x - 200000 + 25x^2 - 500x = 10000x$

or, $x^2 - 20x = 8000$, etc.

37. Let x be the number of men in the regiment; y the number in the side of the square at the first attempt.

Then $y^2 = x - 39$ and $(y+1)^2 = x + 50$.

Hence $(y+1)^2 - y^2 = 89$; whence $2y = 88$, $y = 44$, etc.

XCIX.

1. $x = \frac{29 - 7y}{5} = 5 - y + \frac{4 - 2y}{5}$; let $\frac{4-2y}{5} = m$

$y = \frac{4 - 5m}{2} = 2 - 2m - \frac{m}{2}$; let $\frac{m}{2} = n$; $m = 2n$

Then $y = 2 - 4n - n = 2 - 5n$

and $x = 5 - y + m = 5 - 2 + 5n + 2n = 3 + 7n$

when $n = 0$; $y = 2$ and $x = 3$.

2. $x=\dfrac{92-19y}{7}=13-2y-\dfrac{5y-1}{7}$; let $\dfrac{5y-1}{7}=m$

$y=\dfrac{7m+1}{5}=m+\dfrac{2m+1}{5}$; let $\dfrac{2m+1}{5}=n$

$m=\dfrac{5n-1}{2}=2n+\dfrac{n-1}{2}$; let $\dfrac{n-1}{2}=p$

$n=2p+1$

Then $y=m+n=2n+p+n=4p+2+p+2p+1=7p+3$

$x=13-14p-6-4p-2-p=5-19p$

when $p=0$, $x=5$ and $y=3$.

3. $x=\dfrac{1170-19y}{13}=90-y-\dfrac{6y}{13}$; let $\dfrac{6y}{13}=m$

$y=\dfrac{13m}{6}=2m+\dfrac{m}{6}$; let $\dfrac{m}{6}=n$; $m=6n$

Then $y=13n$ and $x=90-19n$

when $n=0$, $y=0$ and $x=90$

$n=1$, $y=13$ and $x=71$, etc.

4. $x=\dfrac{26-5y}{3}=8-y-\dfrac{2y-2}{3}$; let $\dfrac{2y-2}{3}=m$

$y=\dfrac{3m+2}{2}=m+1+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $y=2n+1+n=3n+1$

$x=8-3n-1-2n=7-5n$

when $n=0$, $y=1$, $x=7$

$n=1$, $y=4$, $x=2$.

5. $5y=14x-7$; $y=2x-1+\dfrac{4x-2}{5}$; let $\dfrac{4x-2}{5}=m$

$x=\dfrac{5m+2}{4}=m+\dfrac{m+2}{4}$; let $\dfrac{m+2}{4}=n$; $m=4n-2$

Then $x=4n-2+n=5n-2$

$y=10n-4-1+4n-2=14n-7$, etc.

6. $x=\dfrac{1031-15y}{11}=93-y-\dfrac{4y-8}{11}$; let $\dfrac{4y-8}{11}=m$

$y=\dfrac{11m+8}{4}=2m+2+\dfrac{3m}{4}$; let $\dfrac{3m}{4}=n$

$m=\dfrac{4n}{3}=n+\dfrac{n}{3}$; let $\dfrac{n}{3}=p$; $n=3p$

Then $y=8p+2+3p=11p+2$

$x=93-11p-2-4p=91-15p$, etc.

7. $y=\dfrac{308-11x}{7}=44-x-\dfrac{4x}{7}$; let $\dfrac{4x}{7}=m$

$x=\dfrac{7m}{4}=m+\dfrac{3m}{4}$; let $\dfrac{3m}{4}=n$

$m=\dfrac{4n}{3}=n+\dfrac{n}{3}$; let $\dfrac{n}{3}=p$; $n=3p$

Then $x=m+n=3p+p+3p=7p$

$y=44-x-m=44-7p-4p=44-11p$, etc.

8. $x=\dfrac{23+19y}{4}=5+4y+\dfrac{3y+3}{4}$; let $\dfrac{3y+3}{4}=m$

$y=\dfrac{4m-3}{3}=m-1+\dfrac{m}{3}$; let $\dfrac{m}{3}=n$; $m=3n$

Then $y=3n-1+n=4n-1$

$x=5+16n-4+3n=19n+1$, etc.

9. $y=\dfrac{20x-683}{9}=2x-75+\dfrac{2x-8}{9}$; let $\dfrac{2x-8}{9}=m$

$x=\dfrac{9m+8}{2}=4m+4+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $x=8n+4+n=9n+4$

$y=18n+8-75+2n=20n-67$, etc.

10. $x=\dfrac{383-7y}{3}=127-2y-\dfrac{y-2}{3}$; let $\dfrac{y-2}{3}=m$

Then $y=3m+2$

$x=127-6m-4-m=123-7m$, etc.

11. $y=\frac{54-27x}{4}=13-6x-\frac{3x-2}{4}$; let $\frac{3x-2}{4}=m$

$x=\frac{4m+2}{3}=m+\frac{m+2}{3}$; let $\frac{m+2}{3}=n$; $m=3n-2$

Then $x=3n-2+n=4n-2$

$y=13-24n+12-3n+2=27-27n$, etc.

12. $x=\frac{653-9y}{7}=93-y-\frac{2y-2}{7}$; let $\frac{2y-2}{7}=m$

$y=\frac{7m+2}{2}=3m+1+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=6n+1+n=7n+1$

$x=93-7n-1-2n=92-9n$, etc.

13. $\frac{x}{7}+\frac{y}{9}=\frac{57}{63}$; $9x+7y=57$

$y=\frac{57-9x}{7}=8-x-\frac{2x-1}{7}$; let $\frac{2x-1}{7}=m$

$x=\frac{7m+1}{2}=3m+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=6n-3+n=7n-3$

$y=8-7n+3-2n+1=12-9n$

when $n=1$, $x=4$, $y=3$.

14. $\frac{x}{11}-\frac{y}{13}=\frac{82}{143}$; $13x-11y=82$

$y=\frac{13x-82}{11}=x-7+\frac{2x-5}{11}$; let $\frac{2x-5}{11}=m$

$x=\frac{11m+5}{2}=5m+2+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=10n-5+2+n=11n-3$

$y=11n-3-7+2n-1=13n-11$

when $n=1$, $x=8$, $y=2$.

15. Let x be the number of florins, y the number of half-crowns; then

$4x+5y=58$; $x=14-y-\frac{y-2}{4}$; let $\frac{y-2}{4}=m$

Then $y=4m+2$ and $x=12-5m$, etc.

16. Let x be the number of half-guineas, y the number of half-crowns; then

$21x+5y=800$; $y=160-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$ and $y=160-21m$

when $m=1$, $x=5$, $y=139$, etc.

7. Let x be the number; y the first quotient, z the second quotient.

Then $\frac{x}{5}=y+\frac{2}{5}$; $\frac{x}{9}=z+\frac{3}{9}$

Hence $5y+2=9z+3$; $y=z+\frac{4z+1}{5}$; let $\frac{4z+1}{5}=m$

$z=\frac{5m-1}{4}=m+\frac{m-1}{4}$; let $\frac{m-1}{4}=n$; $m=4n+1$

$z=5n+1$; $y=9n+2$

Hence $y=2, 11 \ldots$ and hence $x=12, 57 \ldots$

8. Let x be the number of guineas, and y the number of crowns.

Then $21x+5y=235$; $y=47-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$; and $y=47-21m$, etc.

9. Let x be the number of half-guineas, and y the number of half-crowns.

Then $21x+5y=183$; $y=36-4x-\frac{x-3}{5}$ let $\frac{x-3}{5}=m$

Then $x=5m+3$; and $y=36-20m-12-m=24-21m$, etc.

20. Divide by 17; then $19x-31y=\frac{1000}{17}$; and the right-hand side being a fraction, no integral values of x and y can make the left side equal to the right.

21. Let x be the number of oxen, y of sheep, z of hens.

Then $x+y+z=100$, and $100x+20y+z=2000$,

$\therefore 99x+19y=1900$

Hence $y=100-5x-\frac{4x}{19}$; let $\frac{4x}{19}=m$

$x=\frac{19m}{4}=5m-\frac{m}{4}$; let $\frac{m}{4}=n$; $m=4n$

Then $x=19n$; $y=100-99n$

If $n=1$, $x=19$, $y=1$, and $\therefore z=80$.

22. Let A give x sixpences and receive y fourpenny pieces.

Then $6x-4y=58$; or $3x-2y=29$

Then $y=x-14+\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

Then $x=2m+1$ and $y=3m-13$, etc.

23. Let x be the number of half-crowns, y of florins, z of shillings.

Then $x+y=4z$, and $\frac{5x}{2}+2y+z=244$

Hence $x+y=4z$, and $10x+8y+4z=976$

$\therefore 11x+9y=976$

Hence $y=108-x-\frac{2x-4}{9}$; let $\frac{2x-4}{9}=m$

$x=4m+2+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $x=9n+2$; $y=108-9n-2-2n=106-11n$

Hence $x=2$, $y=106$, $z=27$, which gives the greatest number of coins possible.

4. Let there be x half-crowns, y florins, and z fourpenny-pieces.

Then $x+y+z=50$, and $30x+24y+4z=1200$.

Hence $26x+20y=1000$.

$\therefore y=50-x-\frac{3x}{10}$; let $\frac{3x}{10}=m$.

Then $x=\frac{10m}{3}=3m+\frac{m}{3}$; let $\frac{m}{3}=n$; $m=3n$.

Hence $x=10n$ and $y=50-13n$

$\therefore x=10, 20, 30$; $y=37, 24, 11$; $z=3, 6, 9$.

5. Let A give x sovereigns, and receive y dollars.

Then $240x-51y=12$; or, $80x-17y=4$.

Hence $y=4x+\frac{12x-4}{17}$; let $\frac{12x-4}{17}=m$

$$x=m+\frac{5m+4}{12};\ \text{let}\ \frac{5m+4}{12}=n$$

$$m=2n+\frac{2n-4}{5};\ \text{let}\ \frac{2n-4}{5}=p$$

$$n=2p+2+\frac{p}{2};\ \text{let}\ \frac{p}{2}=q;\ p=2q.$$

Then $x=m+n=3n+p=6p+6+5q=17q+6$

$y=4x+m=68q+24+2n+p=82q+28$.

Hence when $q=0$, $x=6$, $y=28$.

5. Let $2x$ and $3y$ be the parts; then $2x+3y=25$.

Then $x=12-y-\frac{y-1}{2}$; let $\frac{y-1}{2}=m$

$y=2m+1$ and $x=11-3m$

When $m=0$, $y=1$, $x=11$; $2x=22$, $3y=3$, etc.

7. Let there be x crowns and y florins; then $5x+2y=49$.

Then $y=24-2x-\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

$x=2m+1$, and $y=22-5m$

Hence we get positive values for x and y, when $m=0, 1, 2, 3, 4$.

7.
$$\begin{array}{l}
x^{2n}-x^ny^n+y^{2n}\\
\underline{x^{2n}+x^ny^n+y^{2n}}\\
x^{4n}-x^{3n}y^n+x^{2n}y^{2n}\\
\qquad +x^{3n}y^n-x^{2n}y^{2n}+x^ny^{3n}\\
\qquad\qquad +x^{2n}y^{2n}-x^ny^{3n}+y^{4n}\\
\hline
x^{4n}\qquad +x^{2n}y^{2n}\qquad +y^{4n}
\end{array}$$

8.
$$\begin{array}{l}
a^{p^2+p}-b^{p^2}+c^p\\
\underline{a^{p^2-p}+b^{1-p^2}+c^{1-p}}\\
a^{2p^2}-a^{p^2-p}b^{p^2}+a^{p^2-p}c^p\\
\qquad +a^{p^2+p}b^{1-p^2}-b+b^{1-p^2}c^p\\
\qquad\qquad +a^{p^2+p}c^{1-p}-b^{p^2}c^{1-p}+c\\
\hline
a^{2p^2}-a^{p^2-p}b^{p^2}+a^{p^2-p}c^p+a^{p^2+p}b^{1-p^2}-b+b^{1-p^2}c^p\\
\qquad\qquad +a^{p^2+p}c^{1-p}-b^{p^2}c^{1-p}+c
\end{array}$$

9.
$$\begin{array}{l}
x^{2p}+x^p+1\\
\underline{x^{2p}+x^p+1}\\
x^{4p}+x^{3p}+x^{2p}\\
\qquad +x^{3p}+x^{2p}+x^p\\
\qquad\qquad +x^{2p}+x^p+1\\
\hline
x^{4p}+2x^{3p}+3x^{2p}+2x^p+1
\end{array}$$

10.
$$\begin{array}{l}
x^{2p}-x^p+1\\
\underline{x^{2p}-x^p+1}\\
x^{4p}-x^{3p}+x^{2p}\\
\qquad -x^{3p}+x^{2p}-x^p\\
\qquad\qquad +x^{2p}-x^p+1\\
\hline
x^{4p}-2x^{3p}+3x^{2p}-2x^p+1
\end{array}$$

CII.

1.
$$\begin{array}{l}
x^m-y^m)\,x^{4m}-y^{4m}\,(x^{3m}+x^{2m}y^m+x^my^{2m}+y^{3m}\\
\qquad\underline{x^{4m}-x^{3m}y^m}\\
\qquad\quad x^{3m}y^m-y^{4m}\\
\qquad\quad \underline{x^{3m}y^m-x^{2m}y^{2m}}\\
\qquad\qquad x^{2m}y^{2m}-y^{4m}\\
\qquad\qquad \underline{x^{2m}y^{2m}-x^my^{3m}}\\
\qquad\qquad\quad x^my^{3m}-y^{4m}\\
\qquad\qquad\quad \underline{x^my^{3m}-y^{4m}}
\end{array}$$

2.
$$
\begin{array}{l}
x^n+y^n)\,x^{5n}+y^{5n}\,(x^{4n}-x^{3n}y^n+x^{2n}y^{2n}-x^ny^{3n}+y^{4n}\\
\qquad \underline{x^{5n}+x^{4n}y^n}\\
\qquad\quad -x^{4n}y^n+y^{5n}\\
\qquad\quad \underline{-x^{4n}y^n-x^{3n}y^{2n}}\\
\qquad\qquad\quad x^{3n}y^{2n}+y^{5n}\\
\qquad\qquad\quad \underline{x^{3n}y^{2n}+x^{2n}y^{3n}}\\
\qquad\qquad\qquad -x^{2n}y^{3n}+y^{5n}\\
\qquad\qquad\qquad \underline{-x^{2n}y^{3n}-x^ny^{4n}}\\
\qquad\qquad\qquad\qquad x^ny^{4n}+y^{5n}\\
\qquad\qquad\qquad\qquad \underline{x^ny^{4n}+y^{5n}}
\end{array}
$$

3.
$$
\begin{array}{l}
x^r-y^r)\,x^{6r}-y^{6r}\,(x^{5r}+x^{4r}y^r,\text{ etc.}\\
\qquad \underline{x^{6r}-x^{5r}y^r}\\
\qquad\qquad x^{5r}y^r-y^{6r},\text{ etc.}
\end{array}
$$

4.
$$
\begin{array}{l}
a^{3p}+b^{2q})\,a^{15p}+b^{10q}\,(a^{12p}-a^{9p}b^{2q},\text{ etc.}\\
\qquad \underline{a^{15p}+a^{12p}b^{2q}}\\
\qquad\qquad -a^{12p}b^{2q}+b^{10q},\text{ etc.}
\end{array}
$$

5.
$$
\begin{array}{l}
x^d-3)\,x^{5d}-243)\,x^{4d}+3x^{3d},\text{ etc.}\\
\qquad \underline{x^{5d}-3x^{4d}}\\
\qquad\qquad 3x^{4d}-243,\text{ etc.}
\end{array}
$$

6.
$$
\begin{array}{l}
a^{2m}+2a^mx^n+4x^{2n})\,a^{4m}+4a^{2m}x^{2n}+16x^{4n}\,(a^{2m}-2a^mx^n+4x^{2n}\\
\qquad\qquad \underline{a^{4m}+2a^{3m}x^n+4a^{2m}x^{2n}}\\
\qquad\qquad\quad -2a^{3m}x^n+16x^{4n}\\
\qquad\qquad\quad \underline{-2a^{3m}x^n-4a^{2m}x^{2n}-8a^mx^{3n}}\\
\qquad\qquad\qquad\qquad 4a^{2m}x^{2n}+8a^mx^{3n}+16x^{4n}\\
\qquad\qquad\qquad\qquad \underline{4a^{2m}x^{2n}+8a^mx^{3n}+16x^{4n}}
\end{array}
$$

7. $$1+5x^p+x^{2p})\,2+9x^p+14x^{3p}+3x^{4p}\,(2-x^p+3x^{2p}$$
$$\underline{2+10x^p+2x^{2p}}$$
$$-x^p-2x^{2p}+14x^{3p}$$
$$\underline{-x^p-5x^{2p}-x^{3p}}$$
$$3x^{2p}+15x^{3p}+3x^{4p}$$
$$\underline{3x^{2p}+15x^{3p}+3x^{4p}}$$

8 $$b^mc^{2m}-2b^{2m}c^m+b^{3m})\,4b^{2m}c^{2m}-13b^{3m}c^{2m}+14b^{4m}c^m-5b^{5m}\,(4b^mc^m-5b^{2m}$$
$$\underline{4b^{2m}c^{2m}-8b^{3m}c^{2m}+4b^{4m}c^m}$$
$$-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}$$
$$\underline{-5b^{3m}c^{2m}+10b^{4m}c^m-5b^{5m}}$$

9. $$a^{6m}+6a^{5m}+15a^{4m}+20a^{3m}+15a^{2m}+6a^m+1\,(a^{3m}+3a^{2m}+3a^m+$$
$$\begin{array}{ll}
 & \underline{a^{6m}} \\
2a^{3m}+3a^{2m} & 6a^{5m}+15a^{4m} \\
 & \underline{6a^{5m}+9a^{4m}} \\
2a^{3m}+6a^{2m}+3a^m & 6a^{4m}+20a^{3m}+15a^{2m} \\
 & \underline{6a^{4m}+18a^{3m}+9a^{2m}} \\
2a^{3m}+6a^{2m}+6a^m+1 & 2a^{3m}+6a^{2m}+6a^m+1 \\
 & \underline{2a^{3m}+6a^{2m}+6a^m+1}
\end{array}$$

10. $$a^{2m}+2a^mb^n+b^{2n}+2a^mc^r+2b^nc^r+c^{2r}\,(a^m+b^n+c^r$$
$$\begin{array}{ll}
 & \underline{a^{2m}} \\
2a^m+b^n & 2a^mb^n+b^{2n} \\
 & \underline{2a^mb^n+b^{2n}} \\
2a^m+2b^n+c^r & 2a^mc^r+2b^nc^r+c^{2r} \\
 & \underline{2a^mc^r+2b^nc^r+c^{2r}}
\end{array}$$

CIII.

$$\begin{array}{l}
x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1 \\
x^{\frac{1}{3}} - 1 \\ \hline
x - 2x^{\frac{2}{3}} + x^{\frac{1}{3}} \\
\quad - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 1 \\ \hline
x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1
\end{array}$$

2.

$$\begin{array}{l}
y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1 \\
y^{\frac{1}{4}} - 1 \\ \hline
y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} \\
\quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1 \\ \hline
y \qquad\qquad\qquad - 1
\end{array}$$

3.

$$\begin{array}{l}
a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}} \\
a^{\frac{2}{3}} - x^{\frac{2}{3}} \\ \hline
a^2 + a^{\frac{4}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{4}{3}} \\
\quad - a^{\frac{4}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2 \\ \hline
a^2 \qquad\qquad\qquad - x^2
\end{array}$$

$$\begin{array}{l}
a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}} \\
a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \\ \hline
a - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}c^{\frac{2}{3}} \\
\quad + a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{2}{3}}c^{\frac{1}{3}} + b^{\frac{1}{3}}c^{\frac{2}{3}} \\
\qquad + a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{2}{3}} + c \\ \hline
a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}
\end{array}$$

$$\begin{array}{l}
5x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y^{\frac{1}{4}} + 3x^{\frac{1}{4}}y^{\frac{1}{2}} + 7y^{\frac{3}{4}} \\
2x^{\frac{1}{4}} - 3y^{\frac{1}{4}} \\ \hline
10x + 4x^{\frac{3}{4}}y^{\frac{1}{4}} + 6x^{\frac{1}{2}}y^{\frac{1}{2}} + 14x^{\frac{1}{4}}y^{\frac{3}{4}} \\
\quad - 15x^{\frac{3}{4}}y^{\frac{1}{4}} - 6x^{\frac{1}{2}}y^{\frac{1}{2}} - 9x^{\frac{1}{4}}y^{\frac{3}{4}} - 21y \\ \hline
10x - 11x^{\frac{3}{4}}y^{\frac{1}{4}} \qquad + 5x^{\frac{1}{4}}y^{\frac{3}{4}} - 21y.
\end{array}$$

$$\begin{array}{l}
m^{\frac{4}{5}} + m^{\frac{3}{5}}n^{\frac{1}{5}} + m^{\frac{2}{5}}n^{\frac{2}{5}} + m^{\frac{1}{5}}n^{\frac{3}{5}} + n^{\frac{4}{5}} \\
m^{\frac{1}{5}} - n^{\frac{1}{5}} \\ \hline
m + m^{\frac{4}{5}}n^{\frac{1}{5}} + m^{\frac{3}{5}}n^{\frac{2}{5}} + m^{\frac{2}{5}}n^{\frac{3}{5}} + m^{\frac{1}{5}}n^{\frac{4}{5}} \\
\quad - m^{\frac{4}{5}}n^{\frac{1}{5}} - m^{\frac{3}{5}}n^{\frac{2}{5}} - m^{\frac{2}{5}}n^{\frac{3}{5}} - m^{\frac{1}{5}}n^{\frac{4}{5}} - n \\ \hline
m \qquad\qquad\qquad\qquad\qquad - n
\end{array}$$

7.
$$
\begin{array}{l}
m^{\frac{2}{3}} - 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\
m^{\frac{2}{3}} + 2d^{\frac{1}{4}}m^{\frac{1}{3}} + 4d^{\frac{1}{2}} \\
\hline
m^{\frac{4}{3}} - 2d^{\frac{1}{4}}m \quad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} \\
\qquad + 2d^{\frac{1}{4}}m \quad - 4d^{\frac{1}{2}}m^{\frac{2}{3}} + 8d^{\frac{3}{4}}m^{\frac{1}{3}} \\
\qquad\qquad\qquad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} - 8d^{\frac{3}{4}}m^{\frac{1}{3}} + 16d \\
\hline
m^{\frac{4}{3}} \qquad\qquad + 4d^{\frac{1}{2}}m^{\frac{2}{3}} \qquad\qquad + 16d
\end{array}
$$

8.
$$
\begin{array}{l}
8a^{\frac{3}{4}} + 4a^{\frac{1}{2}}b^{\frac{1}{4}} + 5a^{\frac{1}{4}}b^{\frac{1}{2}} + 9b^{\frac{3}{4}} \\
2a^{\frac{1}{4}} - 3b^{\frac{1}{4}} \\
\hline
16a + 8a^{\frac{3}{4}}b^{\frac{1}{4}} + 10a^{\frac{1}{2}}b^{\frac{1}{2}} + 18a^{\frac{1}{4}}b^{\frac{3}{4}} \\
\qquad - 24a^{\frac{3}{4}}b^{\frac{1}{4}} - 12a^{\frac{1}{2}}b^{\frac{1}{2}} - 15a^{\frac{1}{4}}b^{\frac{3}{4}} - 27b \\
\hline
16a + 8a^{\frac{3}{4}}b^{\frac{1}{4}} + 10a^{\frac{1}{2}}b^{\frac{1}{2}} + 18a^{\frac{1}{4}}b^{\frac{3}{4}} - 24a^{\frac{3}{4}}b^{\frac{1}{4}}, \text{ etc.}
\end{array}
$$

9.
$$
\begin{array}{l}
x^{\frac{1}{3}} + a^{\frac{1}{3}} \\
x^{\frac{1}{3}} + a^{\frac{1}{3}} \\
\hline
x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} \\
\quad + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} + 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}
\end{array}
$$

10.
$$
\begin{array}{l}
x^{\frac{1}{3}} - a^{\frac{1}{3}} \\
x^{\frac{1}{3}} - a^{\frac{1}{3}} \\
\hline
x^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} \\
\quad - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} - 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}
\end{array}
$$

11.
$$
\begin{array}{l}
x^{\frac{2}{3}} + y^{\frac{2}{3}} \\
x^{\frac{2}{3}} + y^{\frac{2}{3}} \\
\hline
x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} \\
\quad + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} \\
\hline
x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}
\end{array}
$$

12.
$$
\begin{array}{l}
a + b^{\frac{1}{4}} \\
a + b^{\frac{1}{4}} \\
\hline
a^2 + ab^{\frac{1}{4}} \\
\quad + ab^{\frac{1}{4}} + b^{\frac{1}{2}} \\
\hline
a^2 + 2ab^{\frac{1}{4}} + b^{\frac{1}{2}}
\end{array}
$$

13.
$$
\begin{array}{l}
x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\
x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 3 \\
\hline
x - 2x^{\frac{3}{4}} + 3x^{\frac{1}{2}} \\
\quad - 2x^{\frac{3}{4}} + 4x^{\frac{1}{2}} - 6x^{\frac{1}{4}} \\
\qquad + 3x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + 9 \\
\hline
x - 4x^{\frac{3}{4}} + 10x^{\frac{1}{2}} - 12x^{\frac{1}{4}} + 9
\end{array}
$$

14.
$$
\begin{array}{l}
2x^{\frac{2}{5}} + 3x^{\frac{1}{5}} + 4 \\
2x^{\frac{2}{5}} + 3x^{\frac{1}{5}} + 4 \\
\hline
4x^{\frac{4}{5}} + 6x^{\frac{3}{5}} + 8x^{\frac{2}{5}} \\
\quad + 6x^{\frac{3}{5}} + 9x^{\frac{2}{5}} + 12x^{\frac{1}{5}} \\
\qquad + 8x^{\frac{2}{5}} + 12x^{\frac{1}{5}} + 16 \\
\hline
4x^{\frac{4}{5}} + 12x^{\frac{3}{5}} + 25x^{\frac{2}{5}} + 24x^{\frac{1}{5}} + 16
\end{array}
$$

15.

$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}} \\
\underline{x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}} \\
x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{1}{3}} \\
\quad - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} \\
\underline{\qquad\qquad + x^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}} \\
x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}}z^{\frac{1}{3}} - 2y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}
\end{array}$$

16.

$$\begin{array}{l}
x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}} \\
\underline{x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}}} \\
x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} - x^{\frac{1}{4}}z^{\frac{1}{4}} \\
\quad + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} \\
\underline{\qquad\qquad - x^{\frac{1}{4}}z^{\frac{1}{4}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}} \\
x^{\frac{1}{2}} + 4x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2x^{\frac{1}{4}}z^{\frac{1}{4}} - 4y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}
\end{array}$$

CIV.

$$\begin{array}{l}
-y^{\frac{1}{2}})\,x - y\,(x^{\frac{1}{2}} + y^{\frac{1}{2}} \\
\quad \underline{x - x^{\frac{1}{2}}y^{\frac{1}{2}}} \\
\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} - y \\
\qquad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - y}
\end{array}$$

$$\begin{array}{l}
2.\ a^{\frac{1}{2}} + b^{\frac{1}{2}})\,a - b\,(a^{\frac{1}{2}} - b^{\frac{1}{2}} \\
\qquad \underline{a + a^{\frac{1}{2}}b^{\frac{1}{2}}} \\
\qquad\ - a^{\frac{1}{2}}b^{\frac{1}{2}} - b \\
\qquad\ \underline{- a^{\frac{1}{2}}b^{\frac{1}{2}} - b}
\end{array}$$

$$\begin{array}{l}
-y^{\frac{1}{3}})\,x - y\,(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\quad \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} \\
\qquad x^{\frac{2}{3}}y^{\frac{1}{3}} - y \\
\qquad \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}}} \\
\qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
\qquad\qquad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

$$\begin{array}{l}
4.\ a^{\frac{1}{3}} + b^{\frac{1}{3}})\,a + b\,(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\qquad \underline{a + a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
\qquad\ - a^{\frac{2}{3}}b^{\frac{1}{3}} + b \\
\qquad\ \underline{- a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}} \\
\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}} + b \\
\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}} + b}
\end{array}$$

5. $x^{\frac{1}{3}}+y^{\frac{1}{3}})\,x+y\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}$, etc.

$$\underline{x+x^{\frac{2}{3}}y^{\frac{1}{3}}}$$
$$-x^{\frac{2}{3}}y^{\frac{1}{3}}+y, \text{ etc.}$$

6. $m^{\frac{1}{4}}-n^{\frac{1}{4}})\,m-n\,(m^{\frac{3}{4}}+m^{\frac{1}{2}}n^{\frac{1}{4}}$, et

$$\underline{m-m^{\frac{3}{4}}n^{\frac{1}{4}}}$$
$$m^{\frac{3}{4}}n^{\frac{1}{4}}-n, \text{ etc.}$$

7. $x^{\frac{1}{4}}-3y^{\frac{1}{4}})\,x-81y\,(x^{\frac{3}{4}}+3x^{\frac{1}{2}}y^{\frac{1}{4}}$, etc.

$$\underline{x-3x^{\frac{3}{4}}y^{\frac{1}{4}}}$$
$$3x^{\frac{3}{4}}y^{\frac{1}{4}}-81y, \text{ etc.}$$

8. $3a^{\frac{1}{4}}-2b^{\frac{1}{4}})\,81a-16b\,(27a^{\frac{3}{4}}+18a^{\frac{1}{2}}b^{\frac{1}{4}}$, etc.

$$\underline{81a-54a^{\frac{3}{4}}b^{\frac{1}{4}}}$$
$$54a^{\frac{3}{4}}b^{\frac{1}{4}}-16b, \text{ etc.}$$

9. $a^{\frac{1}{2}}+x^{\frac{1}{2}})\,a-x\,(a^{\frac{1}{2}}-x^{\frac{1}{2}}$

$$\underline{a+a^{\frac{1}{2}}x^{\frac{1}{2}}}$$
$$-a^{\frac{1}{2}}x^{\frac{1}{2}}-x$$
$$\underline{-a^{\frac{1}{2}}x^{\frac{1}{2}}-x}$$

10. $m^{\frac{1}{5}}-3)\,m-243\,(m^{\frac{4}{5}}+3m^{\frac{3}{5}}$, e

$$\underline{m-3m^{\frac{4}{5}}}$$
$$3m^{\frac{4}{5}}-243$$
$$\underline{3m^{\frac{4}{5}}-9m^{\frac{3}{5}}}, \text{ etc.}$$

11. $x^{\frac{1}{2}}+7)\,x+17x^{\frac{1}{2}}+70\,(x^{\frac{1}{2}}+10$

$$\underline{x+7x^{\frac{1}{2}}}$$
$$10x^{\frac{1}{2}}+70$$
$$\underline{10x^{\frac{1}{2}}+70}$$

12. $x^{\frac{1}{3}}-3)\,x^{\frac{2}{3}}+x^{\frac{1}{3}}-12\,(x^{\frac{1}{3}}+$

$$\underline{x^{\frac{2}{3}}-3x^{\frac{1}{3}}}$$
$$4x^{\frac{1}{3}}-12$$
$$\underline{4x^{\frac{1}{3}}-12}$$

13. $b^{\frac{1}{3}}-1)\,-b^{\frac{4}{3}}+3b-3b^{\frac{2}{3}}+b^{\frac{1}{3}}\,(-b+2b^{\frac{2}{3}}-b^{\frac{1}{3}}$

$$\underline{-b^{\frac{4}{3}}+b}$$
$$2b-3b^{\frac{2}{3}}$$
$$\underline{2b-2b^{\frac{2}{3}}}$$
$$-b^{\frac{2}{3}}+b^{\frac{1}{3}}$$
$$\underline{-b^{\frac{2}{3}}+b^{\frac{1}{3}}}$$

14. $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}})\,x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}\,(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} - x^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}$

$$x + x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{2}{3}}z^{\frac{1}{3}}$$

$$- x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{2}{3}}z^{\frac{1}{3}} - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + y + z$$
$$- x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$$

$$x^{\frac{1}{3}}y^{\frac{2}{3}} + y - x^{\frac{2}{3}}z^{\frac{1}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + z$$
$$x^{\frac{1}{3}}y^{\frac{2}{3}} + y + y^{\frac{2}{3}}z^{\frac{1}{3}}$$

$$- x^{\frac{2}{3}}z^{\frac{1}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{2}{3}}z^{\frac{1}{3}} + z$$
$$- x^{\frac{2}{3}}z^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{2}{3}}$$

$$- x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{2}{3}}z^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{2}{3}} + z$$
$$- x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{2}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{2}{3}}$$

$$x^{\frac{1}{3}}z^{\frac{2}{3}} + y^{\frac{1}{3}}z^{\frac{2}{3}} + z$$
$$x^{\frac{1}{3}}z^{\frac{2}{3}} + y^{\frac{1}{3}}z^{\frac{2}{3}} + z$$

15. $x^{\frac{1}{3}} + 4)\,x - 5x^{\frac{2}{3}} - 46x^{\frac{1}{3}} - 40\,(x^{\frac{2}{3}} - 9x^{\frac{1}{3}} - 10$

$$x + 4x^{\frac{2}{3}}$$

$$- 9x^{\frac{2}{3}} - 46x^{\frac{1}{3}}$$
$$- 9x^{\frac{2}{3}} - 36x^{\frac{1}{3}}$$

$$- 10x^{\frac{1}{3}} - 40$$
$$- 10x^{\frac{1}{3}} - 40$$

16. $m^{\frac{1}{2}} - m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}})\,m + m^{\frac{1}{2}}n^{\frac{1}{2}} + n\,(m^{\frac{1}{2}} + m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}}$

$$m - m^{\frac{3}{4}}n^{\frac{1}{4}} + m^{\frac{1}{2}}n^{\frac{1}{2}}$$

$$m^{\frac{3}{4}}n^{\frac{1}{4}} + n$$
$$m^{\frac{3}{4}}n^{\frac{1}{4}} - m^{\frac{1}{2}}n^{\frac{1}{2}} + m^{\frac{1}{4}}n^{\frac{3}{4}}$$

$$m^{\frac{1}{2}}n^{\frac{1}{2}} - m^{\frac{1}{4}}n^{\frac{3}{4}} + n$$
$$m^{\frac{1}{2}}n^{\frac{1}{2}} - m^{\frac{1}{4}}n^{\frac{3}{4}} + n$$

17. $$\begin{array}{l} p^{\frac{1}{2}}-2p^{\frac{1}{4}}+1)\,p-4p^{\frac{3}{4}}+6p^{\frac{1}{2}}-4p^{\frac{1}{4}}+1\,(p^{\frac{1}{2}}-2p^{\frac{1}{4}}+1 \\ \qquad\qquad\quad \underline{p-2p^{\frac{3}{4}}+p^{\frac{1}{2}}} \\ \qquad\qquad\qquad -2p^{\frac{3}{4}}+5p^{\frac{1}{2}}-4p^{\frac{1}{4}} \\ \qquad\qquad\qquad \underline{-2p^{\frac{3}{4}}+4p^{\frac{1}{2}}-2p^{\frac{1}{4}}} \\ \qquad\qquad\qquad\qquad\qquad p^{\frac{1}{2}}-2p^{\frac{1}{4}}+1 \\ \qquad\qquad\qquad\qquad\qquad \underline{p^{\frac{1}{2}}-2p^{\frac{1}{4}}+1} \end{array}$$

18. $$\begin{array}{l} 2x^{\frac{1}{2}}+3y^{\frac{1}{2}}+z^{\frac{1}{2}})\,2x+x^{\frac{1}{2}}y^{\frac{1}{2}}-3y-4y^{\frac{1}{2}}z^{\frac{1}{2}}-x^{\frac{1}{2}}z^{\frac{1}{2}}-z\,(x^{\frac{1}{2}}-y^{\frac{1}{2}}-z^{\frac{1}{2}} \\ \qquad\qquad\quad \underline{2x+3x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{\frac{1}{2}}z^{\frac{1}{2}}} \\ \qquad\qquad\qquad -2x^{\frac{1}{2}}y^{\frac{1}{2}}-3y-4y^{\frac{1}{2}}z^{\frac{1}{2}}-2x^{\frac{1}{2}}z^{\frac{1}{2}}-z \\ \qquad\qquad\qquad \underline{-2x^{\frac{1}{2}}y^{\frac{1}{2}}-3y-y^{\frac{1}{2}}z^{\frac{1}{2}}} \\ \qquad\qquad\qquad\qquad -2x^{\frac{1}{2}}z^{\frac{1}{2}}-3y^{\frac{1}{2}}z^{\frac{1}{2}}-z \\ \qquad\qquad\qquad\qquad \underline{-2x^{\frac{1}{2}}z^{\frac{1}{2}}-3y^{\frac{1}{2}}z^{\frac{1}{2}}-z} \end{array}$$

19. $$\begin{array}{l} x^{\frac{4}{5}}-x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}-x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}})\,x+y\,(x^{\frac{1}{5}}+y^{\frac{1}{5}} \\ \qquad\qquad\qquad\qquad \underline{x-x^{\frac{4}{5}}y^{\frac{1}{5}}+x^{\frac{3}{5}}y^{\frac{2}{5}}-x^{\frac{2}{5}}y^{\frac{3}{5}}+x^{\frac{1}{5}}y^{\frac{4}{5}}} \\ \qquad\qquad\qquad\qquad\quad x^{\frac{4}{5}}y^{\frac{1}{5}}-x^{\frac{3}{5}}y^{\frac{2}{5}}+x^{\frac{2}{5}}y^{\frac{3}{5}}-x^{\frac{1}{5}}y^{\frac{4}{5}}+y \\ \qquad\qquad\qquad\qquad\quad \underline{x^{\frac{4}{5}}y^{\frac{1}{5}}-x^{\frac{3}{5}}y^{\frac{2}{5}}+x^{\frac{2}{5}}y^{\frac{3}{5}}-x^{\frac{1}{5}}y^{\frac{4}{5}}+y} \end{array}$$

CV.

1. $$\begin{array}{lll} a^{-1}+b^{-1} & & \\ \underline{a^{-1}-b^{-1}} & & \\ a^{-2}+a^{-1}b^{-1} & & \\ \qquad\;\; \underline{-a^{-1}b^{-1}-b^{-2}} & & \\ a^{-2} \qquad\qquad\quad -b^{-2} & & \end{array}$$

2. $$\begin{array}{l} x^{-3}+b^{-2} \\ \underline{x^{-3}-b^{-2}} \\ x^{-6}+x^{-3}b^{-2} \\ \qquad\;\; \underline{-x^{-3}b^{-2}-b^{-4}} \\ x^{-6} \qquad\qquad\quad -b^{-4} \end{array}$$

$+x+x^{-1}+x^{-3}$

$\cdot x^{-1}$

$- x^2+1+x^{-2}$

$\cdot x^2-1-x^{-2}-x^{-4}$

$-x^{-4}$

4. x^2-1+x^{-2}

x^2+1+x^{-2}

x^4-x^2+1

$+x^2-1+x^{-2}$

$+1-x^{-2}+x^{-4}$

$x^4 \quad +1+x^{-4}$

$+b^{-2}$

$-b^{-2}$

$+a^{-2}b^{-2}$

$-a^{-2}b^{-1}-b^{-4}$

$-b^{-4}$

6. $a^{-1}-b^{-1}+c^{-1}$

$a^{-1}+b^{-1}+c^{-1}$

$a^{-2}-a^{-1}b^{-1}+a^{-1}c^{-1}$

$+a^{-1}b^{-1}-b^{-2}+b^{-1}c^{-1}$

$+a^{-1}c^{-1}-b^{-1}c^{-1}+c^{-2}$

$a^{-2}+2a^{-1}c^{-1}-b^{-2}+c^{-2}$

7. $1+ab^{-1}+a^2b^{-2}$

$1-ab^{-1}+a^2b^{-2}$

$1+ab^{-1}+a^2b^{-2}$

$-ab^{-1}-a^2b^{-2}-a^3b^{-3}$

$+a^2b^{-2}+a^3b^{-3}+a^4b^{-4}$

$1 \quad +a^2b^{-2}+a^4b^{-4}$

8. $a^2b^{-2}+2+a^{-2}b^2$

$a^2b^{-2}-2-a^{-2}b^2$

$a^4b^{-4}+2a^2b^{-2}+1$

$-2a^2b^{-2}-4-2a^{-2}b^2$

$-1-2a^{-2}b^2-a^{-4}b^4$

$a^4b^{-4} \cdot 4 \quad -a^{-4}b^4-4a^{-2}b^2$

9.
$$\begin{array}{l}
4x^{-3}+3x^{-2}+2x^{-1}+1 \\
x^{-2} - x^{-1}+1 \\
\hline
4x^{-5}+3x^{-4}+2x^{-3}+x^{-2} \\
\quad -4x^{-4}-3x^{-3}-2x^{-2}-x^{-1} \\
\qquad +4x^{-3}+3x^{-2}+2x^{-1}+1 \\
\hline
4x^{-5}- x^{-4}+3x^{-3}+2x^{-2}+ x^{-1}+1
\end{array}$$

10.
$$\begin{array}{l}
\frac{5}{2}x^{-2}+3x^{-1}-\frac{7}{3} \\
2x^{-2}-x^{-1}-\frac{1}{2} \\
\hline
5x^{-4}+6x^{-3}-\frac{14}{3}x^{-2} \\
\quad -\frac{5}{2}x^{-3}-3x^{-2}+\frac{7}{3}x^{-1} \\
\qquad -\frac{5}{4}x^{-2}-\frac{3}{2}x^{-1}+\frac{7}{6} \\
\hline
5x^{-4}+\frac{7}{2}x^{-3}-\frac{107}{12}x^{-2}+\frac{5}{6}x^{-1}+\frac{7}{6}
\end{array}$$

CVI.

1.
$$\begin{array}{l}
x+x^{-1})x^3-x^{-3}(x-x^{-1} \\
\qquad x^3+1 \\
\hline
\qquad -1-x^{-3} \\
\qquad -1-x^{-3} \\
\hline
\end{array}$$

2.
$$\begin{array}{l}
a-b^{-1})a^2- b^{-2}(a+b^{-1} \\
\qquad a^2-ab^{-1} \\
\hline
\qquad ab^{-1}-b^{-2} \\
\qquad ab^{-1}-b^{-2} \\
\hline
\end{array}$$

3.
$$\begin{array}{l}
m+n^{-1})m^3+n^{-3}(m^2-mn^{-1}+n^{-2} \\
\qquad m^3+m^2n^{-1} \\
\hline
\qquad -m^2n^{-1}+n^{-3} \\
\qquad -m^2n^{-1}-mn^{-2} \\
\hline
\qquad mn^{-2}+n^{-3} \\
\qquad mn^{-2}+n^{-3} \\
\hline
\end{array}$$

4. $c-d^{-1})c^5-d^{-5}(c^4+c^3d^{-1}+c^2d^{-2}$, etc.

$\underline{c^5-c^4d^{-1}}$

$c^4d^{-1}-d^{-5}$

$\underline{c^4d^{-1}-c^3d^{-2}}$

$c^3d^{-2}-d^{-5}$

$\underline{c^3d^{-2}-c^2d^{-3}}$, etc.

5. $xy^{-1}+x^{-1}y)x^2y^{-2}+2+x^{-2}y^2(xy^{-1}+x^{-1}y$

$\underline{x^2y^{-2}+1}$

$1+x^{-2}y^2$

$\underline{1+x^{-2}y^2}$

6. $a^{-2}-a^{-1}b^{-1}+b^{-2})a^{-4}+a^{-2}b^{-2}+b^{-4}(a^{-2}+a^{-1}b^{-1}+b^{-2}$

$\underline{a^{-4}-a^{-3}b^{-1}+a^{-2}b^{-2}}$

$a^{-3}b^{-1}+b^{-4}$

$\underline{a^{-3}b^{-1}-a^{-2}b^{-2}+a^{-1}b^{-3}}$

$a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}$

$\underline{a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}}$

7. $xy^{-1}-x^{-1}y)x^3y^{-3}-3xy^{-1}+3x^{-1}y-x^{-3}y^3(x^2y^{-2}-2+x^{-2}y^2$

$\underline{x^3y^{-3}-\ \ xy^{-1}}$

$-2xy^{-1}+3x^{-1}y$

$\underline{-2xy^{-1}+2x^{-1}y}$

$x^{-1}y-x^{-3}y^3$

$\underline{x^{-1}y-x^{-3}y^3}$

8. $\dfrac{x^{-2}}{2} - x^{-1} + 3 \Big) \dfrac{3x^{-5}}{4} - 4x^{-4} + \dfrac{77x^{-3}}{8} - \dfrac{43x^{-2}}{4} - \dfrac{33x^{-1}}{4} + 27 \Big(\dfrac{3x^{-3}}{2} - 5x^{-2} + \dfrac{x^{-1}}{4} + 9$

$$\frac{3x^{-5}}{4} - \frac{3x^{-4}}{2} + \frac{9x^{-3}}{2}$$

$$-\frac{5x^{-4}}{2} + \frac{41x^{-3}}{8} - \frac{43x^{-2}}{4}$$

$$-\frac{5x^{-4}}{2} + 5x^{-3} - 15x^{-2}$$

$$\frac{x^{-3}}{8} + \frac{17x^{-2}}{4} - \frac{33x^{-1}}{4}$$

$$\frac{x^{-3}}{8} - \frac{x^{-2}}{4} + \frac{3x^{-1}}{4}$$

$$\frac{9x^{-2}}{2} - 9x^{-1} + 27$$

$$\frac{9x^{-2}}{2} - 9x^{-1} + 27$$

9. $ab^{-1} + a^{-1}b) a^3b^{-3} + a^{-3}b^3 (a^2b^{-2} - 1 + a^{-2}b^2$

$$a^3b^{-3} + ab^{-1}$$

$$-ab^{-1} + a^{-3}b^3$$

$$-ab^{-1} - a^{-1}b$$

$$a^{-1}b + a^{-3}b^3$$

$$a^{-1}b + a^{-3}b^3$$

$$\ldots{}^{-1}+b^{-1}+c^{-1})a^{-3}+b^{-3}+c^{-3}-3a^{-1}b^{-1}c^{-1}(a^{-2}-a^{-1}b^{-1}-a^{-1}c^{-1}+b^{-2}-b^{-1}c^{-1}+c^{-2}$$

$$\ldots{}^{-3}+a^{-2}b^{-1}+a^{-2}c^{-1}$$

$$-a^{-2}b^{-1}-a^{-2}c^{-1}-3a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}$$

$$-a^{-2}b^{-1}-a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}$$

$$-a^{-2}c^{-1}+a^{-1}b^{-2}-2a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3}$$

$$-a^{-2}c^{-1}-a^{-1}b^{-1}c^{-1}-a^{-1}c^{-2}$$

$$a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}+b^{-3}+c^{-3}$$

$$a^{-1}b^{-2}+b^{-3}+b^{-2}c^{-1}$$

$$-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}-b^{-2}c^{-1}+c^{-3}$$

$$-a^{-1}b^{-1}c^{-1}-b^{-2}c^{-1}-b^{-1}c^{-2}$$

$$a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}$$

$$a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}$$

CVII.

1. $x^{\frac{3}{4}}+2x^{\frac{1}{2}}y^{\frac{1}{2}}+2y)\,x^{\frac{3}{2}}-4xy+4x^{\frac{3}{4}}y+4y^2\,(x^{\frac{3}{4}}-2x^{\frac{1}{2}}y^{\frac{1}{2}}+2y$

$$x^{\frac{3}{2}}+2x^{\frac{5}{4}}y^{\frac{1}{2}}+2x^{\frac{3}{4}}y$$

$$-2x^{\frac{5}{4}}y^{\frac{1}{2}}-4xy+2x^{\frac{3}{4}}y$$

$$-2x^{\frac{5}{4}}y^{\frac{1}{2}}-4xy-4x^{\frac{1}{2}}y^{\frac{3}{2}}$$

$$2x^{\frac{3}{4}}y+4x^{\frac{1}{2}}y^{\frac{3}{2}}+4y^2$$

$$2x^{\frac{3}{4}}y+4x^{\frac{1}{2}}y^{\frac{3}{2}}+4y^2$$

2. $\{x^{15ab}\times x^{12}\}^{\frac{1}{3a-2}}=x^{\frac{15ab+12}{3a-2}}$.

3. $(x^{10b+18a})^{\frac{1}{3a-2}}=x^{\frac{10b+18a}{3a-2}}$.

4. $\left\{\dfrac{x^2+a^2-x^2+a^2}{x^4-a^4}-\dfrac{a(x-a-x-a)}{x^4-a^4}\right\}=\left(\dfrac{4a^2}{x^4-a^4}\right)^{\frac{1}{2}}$, etc.

5. $$\begin{array}{l} \frac{7}{3}x^{-2}+4x^{-1}-\frac{2}{7} \\ 3x^{-2}-2x^{-1}-\frac{1}{2} \\ \hline 7x^{-4}+12x^{-3}-\frac{6}{7}x^{-2} \\ \qquad -\frac{14}{3}x^{-3}-8x^{-2}+\frac{4}{7}x^{-1} \\ \qquad\qquad -\frac{7}{6}x^{-2}-2x^{-1}+\frac{1}{7} \\ \hline 7x^{-4}+\frac{22}{3}x^{-3}-\frac{421}{42}x^{-2}-\frac{10}{7}x^{-1}+\frac{1}{7} \end{array}$$

6. $$\frac{x^{a+b+a-b+c-2a}}{x^{c-a}}=x^{c-c+a}=x^{a}.$$

7. $$\begin{array}{l} x^n+y^n)\,x^{2n}-y^{2n}\,(x^n-y^n \\ \qquad x^{2n}+x^ny^n \\ \qquad\qquad -x^ny^n-y^{2n} \\ \qquad\qquad -x^ny^n-y^{2n} \end{array}$$

8. $$\begin{array}{l} a^{\frac{3}{2}}+3ab^{\frac{3}{4}}+3a^{\frac{1}{2}}b^{\frac{3}{2}}+b^{\frac{9}{4}} \\ a^{\frac{1}{2}}-b^{\frac{3}{4}} \\ \hline a^2+3a^{\frac{3}{2}}b^{\frac{3}{4}}+3ab^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{9}{4}} \\ \quad -\ a^{\frac{3}{2}}b^{\frac{3}{4}}-3ab^{\frac{3}{2}}-3a^{\frac{1}{2}}b^{\frac{9}{4}}-b^{3} \\ \hline a^2+2a^{\frac{3}{2}}b^{\frac{3}{4}} \qquad -2a^{\frac{1}{2}}b^{\frac{9}{4}}-b^{3} \end{array}$$

9. $$\begin{array}{l} a^{\frac{1}{3}}-b^{\frac{1}{3}})\,a-b\,(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}, \text{ etc.} \\ \qquad a-a^{\frac{2}{3}}b^{\frac{1}{3}} \\ \hline \qquad\qquad a^{\frac{2}{3}}b^{\frac{1}{3}}-b, \text{ etc.} \end{array}$$

10. $(a^2)^m = a^{2+2+\cdots \;m \text{ terms}} = a^{2m} = a^m \times a^m = (a^m)^2.$

11. $a^{m^n} = a^{mn}$; $\therefore\ m^n = mn$; $m^{n-1} = n$; $m = n^{\frac{1}{n-1}}.$

12. $x^{a+b+c+a+b-c+a-b+c+b+c-a} = x^{2a+2b+2c}.$

13. $(x^p)^p \div (x^p)^{p-q} = x^{p^2} \div x^{p^2-pq} = x^{pq}.$

14. $4a^x \div \dfrac{1}{4a^x} = 4a^x \times 4a^x = 16a^{2x}.$

15. $[\{a^{mn}\}^p] \div [\{a^{mn}\}^{-p}] = a^{mnp} \div a^{-mnp} = a^{2mnp}.$

16. $2a^{2m} + 2a^m b^p - 4a^m c^n - 3a^m b - 3b^{p+1} + 6bc^n.$

17. $a^{m-n+n-m}.\quad b^{n-p+p-n}.\quad c = 1 \times 1 \times c = c.$

18. $$\frac{a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}})}{(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})} = \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$$

19. $(x^{\frac{1}{3}} + x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - x^{\frac{1}{6}} + 1) = x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$

$(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1) = x^{\frac{4}{3}} + x^{\frac{2}{3}} + 1$

20.
$$\begin{array}{l}
a^m - ba^{m-1}x + ca^{m-2}x^2 \\
a^n + ba^{n-1}x - ca^{n-2}x^2 \\
\hline
a^{m+n} - ba^{m+n-1}x + ca^{m+n-2}x^2 \\
\qquad\quad + ba^{m+n-1}x - b^2a^{m+n-2}x^2 + bca^{m+n-3}x^3 \\
\qquad\qquad\qquad\qquad - ca^{m+n-2}x^2 + bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4 \\
\hline
a^{m+n} - b^2a^{m+n-2}x^2 + 2bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4
\end{array}$$

21.
$$\begin{array}{l}
x^{pq-p} + y^{pq-q})\, x^{2pq-2p} - y^{2pq-2q}\, (x^{pq-p} - y^{pq-q} \\
\qquad\qquad\qquad x^{2pq-2p} + x^{pq-p}y^{pq-q} \\
\hline
\qquad\qquad\qquad\qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
\qquad\qquad\qquad\qquad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
\hline
\end{array}$$

22. $\{(a^m)^{\frac{m^2-1}{m}}\}^{\frac{1}{m+1}} = a^{\frac{m^2-1}{m+1}} = a^{m-1}.$

23. $x^{3r} + x^{2r}y^p + x^r y^{2p} + y^{3p}$
$x^r - y^p$

$$\begin{array}{l} x^{4r} + x^{3r}y^p + x^{2r}y^{2p} + x^r y^{3p} \\ \qquad - x^{3r}y^p - x^{2r}y^{2p} - x^r y^{3p} - y^{4p} \\ \hline x^{4r} \qquad\qquad\qquad\qquad\qquad - y^{4p} \end{array}$$

24. $\sqrt[4]{625} = \sqrt{25} = 5$; and $\frac{1}{12^2} = \frac{1}{144}$.

25. $x^{mn-n} - y^{mn-m}$
$x^n - y^m$

$$\overline{x^{mn} - x^n y^{mn-m} - x^{mn-n} y^m + y^{mn}}$$

26. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$
$x^{\frac{1}{3}} - 2x^{-\frac{1}{3}}$

$$\begin{array}{l} x + 3x^{\frac{2}{3}} - x^{\frac{1}{3}} \\ \quad - 2x^{\frac{1}{3}} - 6x^{\frac{1}{3}} + 2x^{-\frac{1}{3}} \\ \hline x + 3x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 7x^{\frac{1}{3}} + 2x^{-\frac{1}{3}} \end{array}$$

CVIII.

1. $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$; $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; $\sqrt[6]{x^3}$ and $\sqrt[6]{y^2}$.

2. $4^{\frac{1}{3}}$ and $2^{\frac{1}{5}}$; $4^{\frac{5}{15}}$ and $2^{\frac{3}{15}}$; $\sqrt[15]{4^5}$ and $\sqrt[15]{2^3}$, etc.

3. $18^{\frac{1}{2}}$ and $50^{\frac{1}{3}}$; $18^{\frac{3}{6}}$ and $50^{\frac{2}{6}}$; $\sqrt[6]{18^3}$ and $\sqrt[6]{50^2}$, etc.

4. $2^{\frac{1}{m}}$ and $2^{\frac{1}{n}}$; $2^{\frac{n}{mn}}$ and $2^{\frac{m}{mn}}$; $\sqrt[mn]{2^n}$ and $\sqrt[mn]{2^m}$.

[illegible] and $b^{\frac{1}{n}}$; $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$; $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$.

[illegible]$)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$; $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$, etc.

CIX.

$\sqrt{}(4\times 6)=2\sqrt{6}$. 2. $\sqrt{50}=\sqrt{}(25\times 2)=5\sqrt{2}$.

$=2a\sqrt{a}$. 4. $\sqrt{}(125a^4d^3)=5a^2d\sqrt{}(5d)$.

$: 2yz^3)=4z\sqrt{}(2yz)$. 6. $\sqrt{}(100\times 10a)=10\sqrt{}(10a)$.

$\times 5c^2)=12c\sqrt{5}$. 8. 7. $\sqrt{}(36\times 11x)=42\sqrt{}(11x)$.

$\frac{1}{9}\times\frac{5}{3}x^3\Big)=6x\sqrt{\frac{5x}{3}}$. 10. $a\sqrt{\left(a^2\times\frac{a}{b}\right)}=a^2\sqrt{\frac{a}{b}}$.

11. $\sqrt{}\{a(a^2+2ax+x^2)\}=(a+x)\sqrt{a}$.

12. $\sqrt{}\{x(x^2-2xy+y^2)\}=(x-y)\sqrt{x}$.

13. $\sqrt{}\{2(25a^2-50ab+25b^2)\}=5(a-b)\sqrt{2}$.

14. $\sqrt{}\{7y(9c^4-6c^2y+y^2)\}=(3c^2-y)\sqrt{}(7y)$.

15. $\sqrt[3]{}(27a^6\times 2b^2)=3a^2\sqrt[3]{}(2b^2)$.

16. $\sqrt[3]{}(20xy\times 8x^3y^6)=2xy^2\sqrt[3]{}(20xy)$.

17. $\sqrt[3]{}(27m^9n^9\times 4n)=3m^3n^3\sqrt[3]{}(4n)$.

18. $\sqrt[3]{}(343a^{15}b^{15}\times 4b)=7a^5b^5\sqrt[3]{}(4b)$.

19. $\sqrt[3]{}\{x(x^3+3x^2y+3xy^2+y^3)\}=(x+y)\sqrt[3]{x}$.

20. $\sqrt[3]{}\{a(a^3-3a^2b+3ab^2-b^3)\}=(a-b)\sqrt[3]{a}$.

CX.

$:\sqrt{16}\,.\sqrt{3}=\sqrt{48}$. 2. $3\sqrt{7}=\sqrt{9}\,.\sqrt{7}=\sqrt{63}$.

$:\sqrt[3]{125}\,.\sqrt[3]{9}=\sqrt[3]{1125}$. 4. $2\sqrt[4]{6}=\sqrt[4]{16}\,.\sqrt[4]{6}=\sqrt[4]{96}$.

5. $3\sqrt[3]{\frac{3}{7}}=\sqrt[3]{27}\,.\sqrt[3]{\frac{3}{7}}=\sqrt[3]{\frac{81}{7}}$.

6. $3\sqrt{a}=\sqrt{9}\,.\sqrt{a}=\sqrt{}(9a)$.

23. $x^{3r}+x^{2r}y^{p}+x^{r}y^{2p}+y^{3p}$

$x^{r}-y^{p}$

$$\begin{array}{l} x^{4r}+x^{3r}y^{p}+x^{2r}y^{2p}+x^{r}y^{3p} \\ \quad -x^{3r}y^{p}-x^{2r}y^{2p}-x^{r}y^{3p}-y^{4p} \\ \hline x^{4r} \qquad\qquad\qquad\qquad\qquad\qquad -y^{4p} \end{array}$$

24. $\sqrt[4]{625}=\sqrt{25}=5$; and $\dfrac{1}{12^2}=\dfrac{1}{144}$.

25. $x^{mn-n}-y^{mn-m}$

$x^{n}-y^{m}$

$$\overline{x^{mn}-x^{n}y^{mn-m}-x^{mn-n}y^{m}+y^{mn}}$$

26. $x^{\frac{3}{4}}+3x^{\frac{1}{2}}-1$

$x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$

$$\begin{array}{l} x+3x^{\frac{3}{4}}-x^{\frac{1}{4}} \\ \quad -2x^{\frac{1}{2}}-6x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \\ \hline x+3x^{\frac{3}{4}}-2x^{\frac{1}{2}}-7x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \end{array}$$

CVIII.

1. $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$; $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; $\sqrt[6]{x^3}$ and $\sqrt[6]{y^2}$.

2. $4^{\frac{1}{3}}$ and $2^{\frac{1}{5}}$; $4^{\frac{5}{15}}$ and $2^{\frac{3}{15}}$; $\sqrt[15]{4^5}$ and $\sqrt[15]{2^3}$, etc.

3. $18^{\frac{1}{2}}$ and $50^{\frac{1}{3}}$; $18^{\frac{3}{6}}$ and $50^{\frac{2}{6}}$; $\sqrt[6]{18^3}$ and $\sqrt[6]{50^2}$, etc.

4. $2^{\frac{1}{m}}$ and $2^{\frac{1}{n}}$; $2^{\frac{n}{mn}}$ and $2^{\frac{m}{mn}}$; $\sqrt[mn]{2^n}$ and $\sqrt[mn]{2^m}$.

5. $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$; $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$; $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$.

6. $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$; $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$, etc.

CIX.

$= \sqrt{(4 \times 6)} = 2\sqrt{6}$. 2. $\sqrt{50} = \sqrt{(25 \times 2)} = 5\sqrt{2}$.

$x^3) = 2a\sqrt{a}$. 4. $\sqrt{(125a^4d^3)} = 5a^2d\sqrt{(5d)}$.

$6 \times 2yz^3) = 4z\sqrt{(2yz)}$. 6. $\sqrt{(100 \times 10a)} = 10\sqrt{(10a)}$.

$44 \times 5c^2) = 12c\sqrt{5}$. 8. 7. $\sqrt{(36 \times 11x)} = 42\sqrt{(11x)}$.

$/\left(\frac{1}{9} \times \frac{5}{3}x^3\right) = 6x\sqrt{\frac{5x}{3}}$. 10. $a\sqrt{\left(a^2 \times \frac{a}{b}\right)} = a^2\sqrt{\frac{a}{b}}$.

11. $\sqrt{\{a(a^2 + 2ax + x^2)\}} = (a + x)\sqrt{a}$.

12. $\sqrt{\{x(x^2 - 2xy + y^2)\}} = (x - y)\sqrt{x}$.

13. $\sqrt{\{2(25a^2 - 50ab + 25b^2)\}} = 5(a - b)\sqrt{2}$.

14. $\sqrt{\{7y(9c^4 - 6c^2y + y^2)\}} = (3c^2 - y)\sqrt{(7y)}$.

15. $\sqrt[3]{(27a^6 \times 2b^2)} = 3a^2\sqrt[3]{(2b^2)}$.

16. $\sqrt[3]{(20xy \times 8x^3y^6)} = 2xy^2\sqrt[3]{(20xy)}$.

17. $\sqrt[3]{(27m^9n^9 \times 4n)} = 3m^3n^3\sqrt[3]{(4n)}$.

18. $\sqrt[3]{(343a^{15}b^{15} \times 4b)} = 7a^5b^5\sqrt[3]{(4b)}$.

19. $\sqrt[3]{\{x(x^3 + 3x^2y + 3xy^2 + y^3)\}} = (x + y)\sqrt[3]{x}$.

20. $\sqrt[3]{\{a(a^3 - 3a^2b + 3ab^2 - b^3)\}} = (a - b)\sqrt[3]{a}$.

CX.

$3 = \sqrt{16}.\sqrt{3} = \sqrt{48}$. 2. $3\sqrt{7} = \sqrt{9}.\sqrt{7} = \sqrt{63}$.

$) = \sqrt[3]{125}.\sqrt[3]{9} = \sqrt[3]{1125}$. 4. $2\sqrt[4]{6} = \sqrt[4]{16}.\sqrt[4]{6} = \sqrt[4]{96}$.

5. $3\sqrt[3]{\frac{3}{7}} = \sqrt[3]{27}.\sqrt[3]{\frac{3}{7}} = \sqrt[3]{\frac{81}{7}}$.

6. $3\sqrt{a} = \sqrt{9}.\sqrt{a} = \sqrt{(9a)}$.

7. $4a\sqrt{(3x)} = \sqrt{(16a^2)}\sqrt{(3x)} = \sqrt{(48a^2x)}$.

8. $2ax\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(4a^2x^2)}\,.\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(3a^3x)}$.

9. $\sqrt{(m+n)^2}.\sqrt{\left(\frac{m-n}{m+n}\right)} = \sqrt{(m^2-n^2)}$.

10. $\sqrt{(a+b)^2}.\sqrt{\left(\frac{1}{a^2-b^2}\right)} = \sqrt{\frac{(a+b)^2}{a^2-b^2}} = \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$.

11. $\sqrt{.}\left(\frac{x-y}{x+y}\right)^2.\sqrt{\left(\frac{x^2+xy}{x^2-2xy+y^2}\right)} = \sqrt{.}\frac{x^2+xy}{(x+y)^2} = \left(\frac{x}{x+y}\right)^{\frac{1}{2}}$.

CXI.

1. $\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$; $\sqrt[3]{4} = \sqrt[6]{4^2} = \sqrt[6]{16}$, etc.

2. $\sqrt{10} = \sqrt[6]{10^3} = \sqrt[6]{1000}$; $\sqrt[3]{15} = \sqrt[6]{15^2} = \sqrt[6]{225}$, etc.

3. $2\sqrt{3} = \sqrt{(4\times3)} = \sqrt{12}$; $3\sqrt{2} = \sqrt{(9\times2)} = \sqrt{18}$, etc.

4. $\sqrt{\frac{3}{5}} = \sqrt[6]{\frac{27}{125}}$; . $\sqrt[3]{\frac{14}{15}} = \sqrt[6]{\frac{196}{225}}$, etc.

5. $3\sqrt{7} = \sqrt{(9\times7)} = \sqrt{63}$; $4\sqrt{3} = \sqrt{(16\times3)} = \sqrt{48}$, etc.

6. $2\sqrt{87} = \sqrt{(4\times87)} = \sqrt{348}$; $3\sqrt{33} = \sqrt{(9\times33)} = \sqrt{297}$, etc.

7. $2\sqrt[3]{22} = \sqrt[3]{176} = \sqrt[6]{30976}$; $3\sqrt[3]{7} = \sqrt[3]{189} = \sqrt[6]{35721}$;
$4\sqrt{2} = \sqrt{32} = \sqrt[6]{32768}$, etc.

8. $3\sqrt{19} = \sqrt{171} = \sqrt[6]{5000211}$; $5\sqrt[3]{18} = \sqrt[3]{2250} = \sqrt[6]{5062500}$;
$3\sqrt[3]{82} = \sqrt[3]{2214} = \sqrt[6]{4901796}$, etc.

9. $2\sqrt[3]{14} = \sqrt[3]{112}$; $5\sqrt[3]{2} = \sqrt[3]{250}$; $3\sqrt[3]{3} = \sqrt[3]{81}$, etc.

10. $\frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{4}}$; $\frac{1}{3}\sqrt{3} = \sqrt{\frac{1}{9}}$; $\frac{1}{4}\sqrt{4} = \sqrt{\frac{1}{16}}$, etc.

CXII.

$\sqrt{3}+8\sqrt{3}+18\sqrt{3}=29\sqrt{3}.$

$0\sqrt{10}+20\sqrt{2}+144\sqrt{2}=30\sqrt{10}+164\sqrt{2}.$

$^{2}\sqrt{x}+b^2\sqrt{x}+c^2\sqrt{x}=(a^2+b^2+c^2)\sqrt{x}.$

$\sqrt[3]{2}+7\sqrt[3]{2}+2\sqrt[3]{2}=13\sqrt[3]{2}.$ 5. $21\sqrt[3]{2}+6\sqrt[3]{2}+6\sqrt[3]{2}=33\sqrt[3]{2}.$

$\sqrt{6}-3\sqrt{6}=\sqrt{6}.$ 7. $9\sqrt{3}-4\sqrt{3}=5\sqrt{3}.$

$2\sqrt{2}-24\sqrt{2}=48\sqrt{2}.$ 9. $10\sqrt[3]{2}-6\sqrt[3]{2}=4\sqrt[3]{2}.$

$1\sqrt[3]{3}-21\sqrt[3]{3}=0.$ 11. $\sqrt{6}\times\sqrt{8}=\sqrt{48}=4\sqrt{3}.$

$/14\times\sqrt{20}=\sqrt{280}=2\sqrt{70}.$ 13. $\sqrt{50}\times\sqrt{200}=\sqrt{10000}=100.$

$/(3a^2b)\times\sqrt[3]{(9ab^2)}=\sqrt[3]{(27a^3b^3)}=3ab.$

$/(12ab)\times\sqrt[3]{(8a^2b^3)}=\sqrt[3]{(96a^3b^4)}=2ab\sqrt[3]{(12b)}.$

$/12\div\sqrt{3}=\sqrt{4}=2.$

$/18\div\sqrt{50}=\sqrt{\frac{9}{25}}=\frac{3}{5}.$

$/(a^2b)\div\sqrt[3]{(ab^2)}=\sqrt[3]{\left(\frac{a}{b}\right)}.$

$/(a^3b)\div\sqrt[4]{(ab^3)}=\sqrt[4]{\left(\frac{a^2}{b^2}\right)}=\sqrt{\left(\frac{a}{b}\right)}.$

$/(x^2+x^3y)\div\sqrt{(x+2x^2y+x^3y^2)}=\sqrt{\left(\frac{x+x^2y}{1+2xy+x^2y^2}\right)}=\sqrt{\frac{x}{1+xy}}.$

CXIII.

$x\times\sqrt{y}=\sqrt{(xy)}.$ 2. $\sqrt{(x-y)}\times\sqrt{y}=\sqrt{(xy-y^2)}.$

$(x+y)\times\sqrt{(x+y)}=x+y.$ 4. $\sqrt{(x-y)}\times\sqrt{(x+y)}=\sqrt{(x^2-y^2)}.$

5. $6\sqrt{x} \times 3\sqrt{x} = 18x$.

6. $7\sqrt{(x+1)} \times 8\sqrt{(x+1)} = 56(x+1)$.

7. $10\sqrt{x} \times 9\sqrt{(x-1)} = 90\sqrt{(x^2-x)}$.

8. $\sqrt{(3x)} \times \sqrt{(4x)} = \sqrt{(12x^2)} = 2x\sqrt{3}$.

9. $\sqrt{x} \times -\sqrt{x} = -x$.

10. $\sqrt{(x-1)} \times -\sqrt{(x-1)} = -(x-1) = 1-x$.

11. $3\sqrt{x} \times -4\sqrt{x} = -12x$.

12. $-2\sqrt{a} \times -3\sqrt{a} = 6a$.

13. $\sqrt{(x-7)} \times -\sqrt{x} = -\sqrt{(x^2-7x)}$.

14. $-2\sqrt{(x+7)} \times -3\sqrt{x} = 6\sqrt{(x^2+7x)}$.

15. $-4\sqrt{(a^2-1)} \times -2\sqrt{(a^2-1)} = 8(a^2-1)$.

16. $2\sqrt{(a^2-2a+3)} \times -3\sqrt{(a^2-2a+3)} = -6(a^2-2a+3) = -6a^2+12a-18$.

CXIV.

1.
$$\begin{array}{l} \sqrt{x}+7 \\ \underline{\sqrt{x}+2} \\ x+7\sqrt{x} \\ \underline{\quad +2\sqrt{x}+14} \\ x+9\sqrt{x}+14 \end{array}$$

2.
$$\begin{array}{l} \sqrt{x}-5 \\ \underline{\sqrt{x}+3} \\ x-5\sqrt{x} \\ \underline{\quad +3\sqrt{x}-15} \\ x-2\sqrt{x}-15 \end{array}$$

3.
$$\begin{array}{l} \sqrt{(a+9)}+3 \\ \underline{\sqrt{(a+9)}-3} \\ a+9+3\sqrt{(a+9)} \\ \underline{\qquad\quad -3\sqrt{(a+9)}-9} \\ a \end{array}$$

4.
$$\begin{array}{l} \sqrt{(a-4)}-7 \\ \underline{\sqrt{(a-4)}+7} \\ a-4-7\sqrt{(a-4)} \\ \underline{\qquad\quad +7\sqrt{(a-4)}-49} \\ a-53 \end{array}$$

$$\begin{array}{l}
3\sqrt{x}-7 \\
\sqrt{x}+4 \\
\hline
3x-7\sqrt{x} \\
\qquad +12\sqrt{x}-28 \\
\hline
3x+5\sqrt{x}-28
\end{array}$$

6. $$\begin{array}{l}
2\sqrt{(x-5)}+4 \\
3\sqrt{(x-5)}-6 \\
\hline
6(x-5)+12\sqrt{(x-5)} \\
\qquad\qquad -12\sqrt{(x-5)}-24 \\
\hline
6x-54
\end{array}$$

7. $$\begin{array}{l}
\sqrt{(6+x)}+\sqrt{x} \\
\sqrt{(6+x)}-\sqrt{x} \\
\hline
6+x+\sqrt{(6x+x^2)} \\
\qquad -\sqrt{(6x+x^2)}-x \\
\hline
6
\end{array}$$

8. $$\begin{array}{l}
\sqrt{(3x+1)}+\sqrt{(2x-1)} \\
\sqrt{3x}\quad -\sqrt{(2x-1)} \\
\hline
\sqrt{(9x^2+3x)}+\sqrt{(6x^2-3x)} \\
\qquad\qquad +\sqrt{(6x^2-x-1)}-(2x-1) \\
\hline
\sqrt{(9x^2+3x)}+\sqrt{(6x^2-3x)}+\sqrt{(6x^2-x-1)}-2x+1
\end{array}$$

9. $$\begin{array}{l}
\sqrt{a}+\sqrt{(a-x)} \\
\sqrt{x}-\sqrt{(a-x)} \\
\hline
\sqrt{(ax)}+\sqrt{(ax-x^2)} \\
\qquad -\sqrt{(a^2-ax)}-(a-x) \\
\hline
\sqrt{(ax)}+\sqrt{(ax-x^2)}-\sqrt{(a^2-ax)}-a+x
\end{array}$$

10. $$\begin{array}{l}
\sqrt{(3+x)}+\sqrt{x} \\
\sqrt{(3+x)} \\
\hline
3+x+\sqrt{(3x+x^2)}
\end{array}$$

11.
$$\begin{array}{l} \sqrt{x}+\sqrt{y}+\sqrt{z} \\ \sqrt{x}-\sqrt{y}+\sqrt{z} \\ \hline x+\sqrt{(xy)}+\sqrt{(xz)} \\ \quad -\sqrt{(xy)}-y-\sqrt{(yz)} \\ \qquad\qquad +\sqrt{(xz)}+\sqrt{(yz)}+z \\ \hline x-y+2\sqrt{(xz)}+z \end{array}$$

12.
$$\begin{array}{l} \sqrt{a}+\sqrt{(a-x)}+\sqrt{x} \\ \sqrt{a}-\sqrt{(a-x)}+\sqrt{x} \\ \hline a+\sqrt{(a^2-ax)}+\sqrt{(ax)} \\ \quad -\sqrt{(a^2-ax)}-(a-x)-\sqrt{(ax-x^2)} \\ \qquad\qquad +\sqrt{(ax)}+\sqrt{(ax-x^2)}+x \\ \hline 2x+2\sqrt{(ax)} \end{array}$$

13.
$$\begin{array}{l} 21+\sqrt{(x^2-9)} \\ 21+\sqrt{(x^2-9)} \\ \hline 441+21\sqrt{(x^2-9)} \\ \qquad +21\sqrt{(x^2-9)}+x^2-9 \\ \hline 432+42\sqrt{(x^2-9)}+x^2 \end{array}$$

14.
$$\begin{array}{l} \sqrt{(x+3)}+\sqrt{(x+8)} \\ \sqrt{(x+3)}+\sqrt{(x+8)} \\ \hline x+3+\sqrt{(x^2+11x+24)} \\ \qquad +\sqrt{(x^2+11x+24)}+x+8 \\ \hline 2x+11+2\sqrt{(x^2+11x+24)} \end{array}$$

15.
$$\begin{array}{l} \sqrt{x}+\sqrt{(x-4)} \\ \sqrt{x}+\sqrt{(x-4)} \\ \hline x+\sqrt{(x^2-4x)} \\ \quad +\sqrt{(x^2-4x)}+x-4 \\ \hline 2x-4+2\sqrt{(x^2-4x)} \end{array}$$

16.
$$\begin{array}{l} \sqrt{(x-6)}+\sqrt{x} \\ \sqrt{(x-6)}+\sqrt{x} \\ \hline x-6+\sqrt{(x^2-6x)} \\ \qquad +\sqrt{(x^2-6x)}+x \\ \hline 2x-6+2\sqrt{(x^2-6x)} \end{array}$$

$$\begin{array}{l}
2\sqrt{x}-3 \\
2\sqrt{x}-3 \\
\hline
4x-6\sqrt{x} \\
\quad -6\sqrt{x}+9 \\
\hline
4x-12\sqrt{x}+9
\end{array}$$

18.
$$\begin{array}{l}
\sqrt{(x+y)}-\sqrt{(x-y)} \\
\sqrt{(x+y)}-\sqrt{(x-y)} \\
\hline
x+y-\sqrt{(x^2-y^2)} \\
\qquad -\sqrt{(x^2-y^2)}+x-y \\
\hline
2x-2\sqrt{(x^2-y^2)}
\end{array}$$

19.
$$\begin{array}{l}
\sqrt{x}\,.\,\sqrt{(x+1)}-\sqrt{(x-1)} \\
\sqrt{x}\,.\,\sqrt{(x+1)}-.\,\sqrt{(x-1)} \\
\hline
x(x+1)-\sqrt{x}\,.\,\sqrt{(x^2-1)} \\
\qquad -\sqrt{x}\,.\,\sqrt{(x^2-1)}+(x-1) \\
\hline
x^2+2x-1-2\sqrt{(x^3-x)}
\end{array}$$

20.
$$\begin{array}{l}
\sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\
\sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\
\hline
x+1+\sqrt{x}.\sqrt{(x^2-1)} \\
\qquad +\sqrt{x}.\sqrt{(x^2-1)}+x(x-1) \\
\hline
x^2+1+2\sqrt{(x^3-x)}
\end{array}$$

CXV.

$c-d=(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$.

2. $c^2-d=(c+\sqrt{d})(c-\sqrt{d})$.

$c-d^2=(\sqrt{c}+d)(\sqrt{c}-d)$.

4. $1-y=(1+\sqrt{y})(1-\sqrt{y})$.

5. $1-3x^2=(1+\sqrt{3}.x)(1-\sqrt{3}.x)$.

6. $5m^2-1=(\sqrt{5}.m+1)(\sqrt{5}.m-1)$.

7. $4a^2-3x=\{2a+\sqrt{(3x)}\}\{2a-\sqrt{(3x)}\}$.

8. $9-8n=\{3+2\sqrt{(2n)}\}\{3-2\sqrt{(2n)}\}$.

9. $11n^2-16=(\sqrt{11}.n+4)(\sqrt{11}.n-4)$.

10. $p^2-4r=(p+2\sqrt{r})(p-2\sqrt{r})$.

11. $p-3q^2=(\sqrt{p}+\sqrt{3}.q)(\sqrt{p}-\sqrt{3}.q)$.

12. $a^{2m} - b^n = (a^m + b^{\frac{n}{2}})(a^m - b^{\frac{n}{2}})$.

13. $\dfrac{1}{a - \sqrt{b}} \times \dfrac{a + \sqrt{b}}{a + \sqrt{b}} = \dfrac{a + \sqrt{b}}{a^2 - b}$.

14. $\dfrac{\sqrt{a}}{\sqrt{a} - \sqrt{b}} \times \dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = \dfrac{a + \sqrt{(ab)}}{a - b}$.

15. $\dfrac{4 + 3\sqrt{2}}{3 - 2\sqrt{2}} \times \dfrac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = \dfrac{12 + 17\sqrt{2} + 12}{9 - 8} = 24 + 17\sqrt{2}$.

16. $\dfrac{2}{2 - \sqrt{2}} \times \dfrac{2 + \sqrt{2}}{2 + \sqrt{2}} = \dfrac{4 + 2\sqrt{2}}{4 - 2} = \dfrac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2}$.

17. $\dfrac{\sqrt{3}}{2 - \sqrt{3}} \times \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{2\sqrt{3} + 3}{4 - 3} = 3 + 2\sqrt{3}$.

18. $\dfrac{2 - \sqrt{2}}{2 + \sqrt{2}} \times \dfrac{2 - \sqrt{2}}{2 - \sqrt{2}} = \dfrac{4 - 4\sqrt{2} + 2}{4 - 2} = \dfrac{6 - 4\sqrt{2}}{2} = 3 - 2\sqrt{2}$.

19. $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}} \times \dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}} = \dfrac{a + 2\sqrt{(ax)} + x}{a - x}$.

20. $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}} \times \dfrac{1 + \sqrt{x}}{1 + \sqrt{x}} = \dfrac{1 + 2\sqrt{x} + x}{1 - x}$.

21. $\dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} \times \dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}}$

$$= \frac{a + x + 2\sqrt{(a^2 - x^2)} + a - x}{a + x - (a - x)} = \frac{2a + 2\sqrt{(a^2 - x^2)}}{2x} = \frac{a + \sqrt{(a^2 - x^2)}}{x}$$

22. $\dfrac{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}{\sqrt{(m^2+1)} + \sqrt{(m^2-1)}} \times \dfrac{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}{\sqrt{(m^2+1)} - \sqrt{(m^2-1)}}$

$$= \frac{m^2 + 1 - 2\sqrt{(m^4 - 1)} + m^2 - 1}{m^2 + 1 - (m^2 - 1)} = \frac{2m^2 - 2\sqrt{(m^4 - 1)}}{2}$$

$$= m^2 - \sqrt{(m^4 - 1)}.$$

23. $\dfrac{a + \sqrt{(a^2 - 1)}}{a - \sqrt{(a^2 - 1)}} \times \dfrac{a + \sqrt{(a^2 - 1)}}{a + \sqrt{(a^2 - 1)}} = \dfrac{a^2 + 2a\sqrt{(a^2 - 1)} + a^2 - 1}{a^2 - (a^2 - 1)}$

$$= 2a^2 - 1 + 2a\sqrt{(a^2 - 1)}.$$

$$\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}\times\frac{a+\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}=\frac{a^2+2a\sqrt{(a^2-x^2)}+a^2-x^2}{a^2-(a^2-x^2)}$$

$$=\frac{2a^2-x^2+2a\sqrt{(a^2-x^2)}}{x^2}.$$

CXVI.

$$\begin{array}{l} \ldots+\sqrt{(-3)} \\ \ldots-\sqrt{(-3)} \\ \hline \ldots+4\sqrt{(-3)} \\ \quad -4\sqrt{(-3)}-(-3) \\ \hline \qquad 19 \end{array}$$

2. $$\begin{array}{l} \sqrt{3}-2\sqrt{(-2)} \\ \sqrt{3}+2\sqrt{(-2)} \\ \hline 3-2\sqrt{(-6)} \\ \quad +2\sqrt{(-6)}-4\times(-2) \\ \hline \qquad 11 \end{array}$$

3. $$\begin{array}{l} 4\sqrt{(-2)}-2\sqrt{2} \\ \tfrac{1}{2}\sqrt{(-2)}-3\sqrt{2} \\ \hline 2\times(-2)-\sqrt{(-4)} \\ \qquad -12\sqrt{(-4)}+6\times 2 \\ \hline \qquad 8-13\sqrt{(-4)}=8-26\sqrt{(-1)} \end{array}$$

4. $$\begin{array}{l} \sqrt{(-2)}+\sqrt{(-3)}+\sqrt{(-4)} \\ \sqrt{(-2)}-\sqrt{(-3)}-\sqrt{(-4)} \\ \hline -2+\sqrt{6}+2\sqrt{2} \\ \quad -\sqrt{6}-(-3)-2\sqrt{3} \\ \qquad -2\sqrt{2}-2\sqrt{3}-(-4) \\ \hline -2+3-4\sqrt{3}+4=5-4\sqrt{3} \end{array}$$

$$\begin{array}{l} \ldots\sqrt{(-a)}+\sqrt{(-b)} \\ \ldots\sqrt{(-a)}-2\sqrt{(-b)} \\ \hline -12a+4\sqrt{(ab)} \\ \qquad -6\sqrt{(ab)}+2b \\ \hline -12a-2\sqrt{(ab)}+2b \end{array}$$

6. $$\begin{array}{l} a+\sqrt{(-a)} \\ a-\sqrt{(-a)} \\ \hline a^2+a\sqrt{(-a)} \\ \quad -a\sqrt{(-a)}+a \\ \hline a^2+a \end{array}$$

7.
$$\begin{array}{l}
a\sqrt{(-a)}+b\sqrt{(-b)} \\
a\sqrt{(-a)}-b\sqrt{(-b)} \\
\hline
-a^3+ab\sqrt{(ab)} \\
\qquad -ab\sqrt{(ab)}+b^3 \\
\hline
-a^3 \qquad\qquad +b^3
\end{array}$$

8.
$$\begin{array}{l}
a+\beta\sqrt{(-1)} \\
a-\beta\sqrt{(-1)} \\
\hline
a^2+a\beta\sqrt{(-1)} \\
\qquad -a\beta\sqrt{(-1)}+\beta^2 \\
\hline
a^2 \qquad\qquad +\beta^2
\end{array}$$

9.
$$\begin{array}{l}
1-\sqrt{(1-e^2)} \\
1+\sqrt{(1-e^2)} \\
\hline
1-\sqrt{(1-e^2)} \\
\quad +\sqrt{(1-e^2)}-(1-e^2) \\
\hline
\qquad\qquad e^2
\end{array}$$

10.
$$\begin{array}{l}
e^{p\sqrt{(-1)}}+e^{-p\sqrt{(-1)}} \\
e^{p\sqrt{(-1)}}-e^{-p\sqrt{(-1)}} \\
\hline
e^{2p\sqrt{(-1)}}+1 \\
\qquad\qquad -1-e^{-2p\sqrt{(-1)}} \\
\hline
e^{2p\sqrt{(-1)}} \qquad -e^{-2p\sqrt{(-1)}}
\end{array}$$

CXVII.

1. $$\frac{3x+3\sqrt{(xy)}-3\sqrt{(xy)}+3y}{9\sqrt{(xy)}}=\frac{x+y}{3\sqrt{(xy)}}.$$

2. $$\{1+2\sqrt{(-1)}+(-1)\}+\{1-2\sqrt{(-1)}+(-1)\}$$
$$=2\sqrt{(-1)}-2\sqrt{(-1)}=0.$$

3. $$\frac{2\sqrt{(xy)}+2y+2x-2\sqrt{(xy)}}{4\sqrt{(xy)}}=\frac{x+y}{2\sqrt{(xy)}}.$$

4. $\{1+2\sqrt{(-1)}-1\}-\{1-2\sqrt{(-1)}-1\}=4\sqrt{(-1)}=\sqrt{(-16)}.$

5.
$$\begin{array}{l}
x^2+\sqrt{2}.ax+a^2)\,x^4+a^4\,(x^2-\sqrt{2}.ax+a^2 \\
\qquad\qquad\qquad x^4+\sqrt{2}.ax^3+a^2x^2 \\
\hline
\qquad\qquad\qquad -\sqrt{2}.ax^3-a^2x^2+a^4 \\
\qquad\qquad\qquad -\sqrt{2}.ax^3-2a^2x^2-\sqrt{2}.a^3x \\
\hline
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\
\hline
\end{array}$$

6. $m^2-\sqrt{2}.mn+n^2)\,m^4+n^4\,(m^2+\sqrt{2}.mn+n^2$

$$\begin{array}{l}
\underline{m^4-\sqrt{2}.m^3n+m^2n^2}\\
\quad\sqrt{2}.m^3n-m^2n^2+n^4\\
\quad\underline{\sqrt{2}.m^3n-2m^2n^2+\sqrt{2}.mn^3}\\
\qquad\qquad m^2n^2-\sqrt{2}.mn^3+n^4\\
\qquad\qquad \underline{m^2n^2-\sqrt{2}.mn^3+n^4}
\end{array}$$

7. $\sqrt{x}.\sqrt{.}(x^2+2xy+y^2)+\sqrt{x}.\sqrt{(x^2-2xy+y^2)}$
$=\sqrt{x}.(x+y)+\sqrt{x}.(x-y)=2x\sqrt{x}.$

8. $$\frac{a\sqrt{a}+a\sqrt{b}-b\sqrt{a}-b\sqrt{b}-(a\sqrt{a}+b\sqrt{a}-a\sqrt{b}-b\sqrt{b})}{a-b}$$
$$=\frac{2a\sqrt{b}-2b\sqrt{a}}{a-b}.$$

9. $a^2.\frac{c}{b}-2a.\sqrt{\frac{c}{b}}.\sqrt{(cd)}+cd=\frac{a^2c}{b}-2ac\sqrt{\frac{d}{b}}+cd.$

10. $a^{2\sqrt{2}}-2a^{\sqrt{2}}.\frac{1}{a^{\sqrt{2}}}+\frac{1}{a^{2\sqrt{2}}}=a^{2\sqrt{2}}-2+\frac{1}{a^{2\sqrt{2}}}.$

11. $$\frac{x^2+a^2+2\sqrt{(x^4-a^4)}+x^2-a^2+x^2+a^2-2\sqrt{(x^4-a^4)}+x^2-a^2}{(x^2+a^2)-(x^2-a^2)}$$
$$=\frac{4x^2}{2a^2}=\frac{2x^2}{a^2}.$$

12. $\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}}\right\}\div\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x^2)}}\right\}=\frac{\sqrt{(1-x^2)}}{\sqrt{(1+x)}}=\sqrt{(1-x)}.$

13. $$\frac{x-1}{x+1}\left\{\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}+\frac{(1+\sqrt{x})(1-\sqrt{x})}{\sqrt{x}(1+\sqrt{x})}\right\}$$
$$=\frac{x-1}{x+1}\left\{\sqrt{x}+1+\frac{1-\sqrt{x}}{\sqrt{x}}\right\}=\frac{x-1}{x+1}\left(\frac{x+\sqrt{x}+1-\sqrt{x}}{\sqrt{x}}\right)=\frac{x-1}{\sqrt{x}}.$$

14. $\frac{x}{4}+3-2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{x}{2}-2\sqrt{\left(\frac{x^2}{16}-9\right)}.$

15. $x+a-2\sqrt{(x^2-a^2)}+x-a=2x-2\sqrt{(x^2-a^2)}.$

16. $\sqrt[m]{(a^{2m-n+n}.b^{5m+1+m-1}.c^{3p+m-3p})}=\sqrt[m]{(a^{2m}.b^{6m}.c^{m})}=a^2b^6c.$

17. $\{-1-a\sqrt{(-1)}\}^2=1+2a\sqrt{(-1)}-a^2$

$$\{1+2a\sqrt{(-1)}-a^2\}^2=1-4a^2+a^4+4a\sqrt{(-1)}-2a^2-4a^3\sqrt{(-1)}$$

$$\begin{array}{l} 1-6a^2+a^4+4a\sqrt{(-1)}-4a^3\sqrt{(-1)} \\ -1-a\sqrt{(-1)} \\ \hline -1+6a^2-a^4-4a\sqrt{(-1)}+4a^3\sqrt{(-1)} \\ \quad -a\sqrt{(-1)}+6a^3\sqrt{(-1)}-a^5\sqrt{(-1)}+4a^2-4a^4 \\ \hline -1+10a^2-5a^4+(10a^3-a^5-5a)\sqrt{(-1)} \end{array}$$

18. $3\sqrt[3]{3}-(-8)+4\sqrt[3]{3}=8+7\sqrt[3]{3}.$

19. $\dfrac{6c^2}{x-1}.\sqrt{\left(\dfrac{x(4x^2-8x+4)}{3c^3}\right)}=\dfrac{6c^2}{x-1}\cdot\dfrac{2x-2}{c}.\sqrt{\dfrac{x}{3c}}=\dfrac{12c\sqrt{x}}{\sqrt{(3c)}}=4\sqrt{(3cx)}.$

20. $\dfrac{x}{x-7}\{\sqrt[3]{[3p^2\{x^3-21x^2+147x-343\}]}\}=\dfrac{x}{x-7}\cdot\sqrt[3]{(3p^2)}.(x-7)$

$=x\sqrt[3]{(3p^2)}.$

21. $2(n-1)\sqrt[3]{\left[\dfrac{1}{2n}\cdot\left(-\dfrac{1}{n^3-3n^2+3n-1}\right)\right]}=2(n-1)\sqrt[3]{\left(\dfrac{1}{2n}\right)\cdot\left(-\dfrac{1}{n-1}\right)}$

$=-2\sqrt[3]{\dfrac{1}{2n}}=-\sqrt[3]{\dfrac{8}{2n}}=-\sqrt[3]{\dfrac{4n^2}{n^3}}=\dfrac{1}{n}\sqrt[3]{(-4n^2)}.$

22. $2(n-1).3\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{2(n-1)\times5}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=6n\sqrt{7}-6\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{10n}{3}\sqrt{7}-\dfrac{10}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=9n\sqrt{7}-10\sqrt{7}.$

23. $\sqrt{(17^2-33)}\quad\sqrt[3]{(65^2-129)}=\sqrt{256}-\sqrt[3]{4096}=16-16=0.$

CXVIII.

Adopt the notation of Art. 316; then

. $\left.\begin{aligned}x+y&=10\\ xy&=21\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=100\\ 4xy&=84\end{aligned}\right\}$; $x^2-2xy+y^2=16$; $x-y=4$, etc.

. $\left.\begin{aligned}x+y&=16\\ xy&=55\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=256\\ 4xy&=220\end{aligned}\right\}$ $x^2-2xy+y^2=36$; $x-y=6$, etc.

. $\left.\begin{aligned}x+y&=9\\ xy&=14\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=81\\ 4xy&=56\end{aligned}\right\}$; $x^2-2xy+y^2=25$; $x-y=5$, etc.

. $\left.\begin{aligned}x+y&=94\\ xy&=2205\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=8836\\ 4xy&=8820\end{aligned}\right\}$; $x^2-2xy+y^2=16$;
$x-y=4$, etc.

$\left.\begin{aligned}x+y&=13\\ xy&=30\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=169\\ 4xy&=120\end{aligned}\right\}$; $x^2-2xy+y^2=49$; $x-y=7$, etc.

$\left.\begin{aligned}x+y&=38\\ xy&=360\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=1444\\ 4xy&=1440\end{aligned}\right\}$; $x^2-2xy+y^2=4$; $x-y=2$, etc.

$\left.\begin{aligned}x+y&=14\\ xy&=24\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=196\\ 4xy&=96\end{aligned}\right\}$; $x^2-2xy+y^2=100$;
$x-y=10$, etc.

$\left.\begin{aligned}x+y&=103\\ xy&=396\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=10609\\ 4xy&=1584\end{aligned}\right\}$; $x^2-2xy+y^2=9025$;
$x-y=95$, etc.

$\left.\begin{aligned}x+y&=75\\ xy&=756\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=5625\\ 4xy&=3024\end{aligned}\right\}$; $x^2-2xy+y^2=2601$;
$x-y=51$, etc.

$\left.\begin{aligned}x+y&=87\\ xy&=1512\end{aligned}\right\}$; $\left.\begin{aligned}x^2+2xy+y^2&=7569\\ 4xy&=6048\end{aligned}\right\}$; $x^2-2xy+y^2=1521$;
$x-y=39$, etc.

11. $\left.\begin{aligned} x+y&=\frac{7}{2}\\ xy&=\frac{10}{4}\end{aligned}\right\}$; $\left.\begin{aligned} x^2+2xy+y^2&=\frac{49}{4}\\ 4xy\quad&=\frac{40}{4}\end{aligned}\right\}$; $x^2-2xy+y^2=\frac{9}{4}$;

$x-y=\frac{3}{2}$, etc.

12. $\left.\begin{aligned} x+y&=57\\ xy&=540\end{aligned}\right\}$ $\left.\begin{aligned} x^2+2xy+y^2&=3249\\ 4xy\quad&=2160\end{aligned}\right\}$; $x^2-2xy+y^2=1089$;

$x-y=33$, etc.

CXIX.

1. $x=7^2=49$.
2. $x=9^2=81$.
3. $x=5^2=25$.
4. $x=2^3=8$.
5. $x=3^3=27$.
6. $x=4^4=256$.
7. $x+9=36$, etc.
8. $x-7=49$, etc.
9. $x-15=64$, etc.
10. $x-9=144$, etc.
11. $4x-16=8$, etc.
12. $3\sqrt{x}=18$; $x=36$.
13. $\sqrt[3]{(2x+3)}=3$; $2x+3=27$; $2x=24$; $x=12$.
14. $c\sqrt{x}=a-b$; $c^2x=(a-b)^2$; $x=\frac{(a-b)^2}{c^2}$.
15. $\sqrt{(x^2-9)}=9-x$; $x^2-9=81-18x+x^2$; $18x=90$; $x=5$.
16. $x^2-11=x^2-2x+1$; $2x=12$; $x=6$.
17. $4x^2+5x-2=4x^2+4x+1$; $x=3$.
18. $9x^2-12x-51=9x^2-18x+9$; $6x=60$; $x=10$.
19. $x^2-ax+b=x^2+2ax+a^2$; $3ax=b-a^2$, etc.
20. $25x^2-3mx+n=m^2+10mx+25x^2$; $13mx=n-m^2$, etc.

CXX.

1. $\sqrt{(16+x)}=8-\sqrt{x}$; $16+x=64-16\sqrt{x}+x$; $16\sqrt{x}=48$; $\sqrt{x}=3$; $x=9$.

2. $x-16=64-16\sqrt{x}+x$; $16\sqrt{x}=80$; $\sqrt{x}=5$; $x=25$.

3. $\sqrt{(x+15)}=15-\sqrt{x}$; $x+15=225-30\sqrt{x}+x$; $30\sqrt{x}=210$; $\sqrt{x}=7$; $x=49$.

4. $x-21=x-2\sqrt{x}+1$; $2\sqrt{x}=22$; $\sqrt{x}=11$; $x=121$.

5. $x-1=x+4-6\sqrt{(x+4)}+9$; $6\sqrt{(x+4)}=14$; $\sqrt{(x+4)}=\dfrac{7}{3}$;

$x+4=\dfrac{49}{9}$; $x=\dfrac{13}{9}$.

6. $1+2\sqrt{(3x+1)}+3x+1=4x+4$; $2\sqrt{(3x+1)}=x+2$;
Squaring both sides, $4(3x+1)=x^2+4x+4$; $x^2=8x$, etc.

7. $1-2\sqrt{(1-3x)}+1-3x=4(1-x)$; $2\sqrt{(1-3x)}=x-2$;
$4(1-3x)=x^2-2x+4$; $x^2+8x=0$, etc.

8. $a-\sqrt{x}=\sqrt{(x-a)}$; $a^2-2a\sqrt{x}+x=x-a$; $2\sqrt{x}=1+a$;
$4x=(1+a)^2$, etc.

9. $\sqrt{(x-m)}=\dfrac{m-2\sqrt{x}}{2}$; $x-m=\dfrac{m^2-4m\sqrt{x}+4x}{4}$; $-4m=m^2-4m\sqrt{x}$;

$\sqrt{x}=\dfrac{m+4}{4}$; $x=\left(\dfrac{m+4}{4}\right)^2$.

10. $\sqrt{(x-4)}=3-\sqrt{(x-1)}$; $x-4=9-6\sqrt{(x-1)}+x-1$;
$6\sqrt{(x-1)}=12$; $\sqrt{(x-1)}=2$; $x-1=4$; $x=5$.

CXXI.

1. $\sqrt{(x^2-9x)}+x-9=36$; $\sqrt{(x^2-9x)}=45-x$;
$x^2-9x=2025-90x+x^2$; $81x=2025$; $x=25$.

2. $x+\sqrt{(x^2-21x)}=35$; $x^2-21x=1225-70x+x^2$; $49x=1225$;
$x=25$.

3. $x+7+\sqrt{(x^2+7x)}=28$; $\sqrt{(x^2+7x)}=21-x$;
$x^2+7x=441-42x+x^2$; $49x=441$; $x=9$.

4. $x-15+\sqrt{(x^2-15x)}=105$; $\sqrt{(x^2-15x)}=120-x$;
$x^2-15x=14400-240x+x^2$; $225x=14400$; $x=64$.

5. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-20x=144$, etc.

6. $\sqrt{(3ax+x^2)}+3a+x-9a=0$; $\sqrt{(3ax+x^2)}=6a-x$;
$3ax+x^2=36a^2-12ax+x^2$; $15ax=36a^2$, etc.

7. $b^2-ax=b^2-ax-ab+bx$; $bx=ab$; $x=a$.

8. $2+\sqrt{x}-x=\dfrac{4+\sqrt{x}}{2}$; $4+2\sqrt{x}-2x=4+\sqrt{x}$; $\sqrt{x}=2x$; $x=$

whence $x=0$ or $\dfrac{1}{4}$.

9. $x+28\sqrt{x}+192=x+36\sqrt{x}+128$; $8\sqrt{x}=64$; $x=64$.

10. $x-6\sqrt{x}-16=x-10\sqrt{x}+24$; $4\sqrt{x}=40$; $x=100$.

CXXII.

1. $x-3\sqrt{x}=4$; $3\sqrt{x}=x-4$; $9x=x^2-8x+16$; $x^2-17x=-$

2. $14\sqrt{x}=x+45$; $196x=x^2+90x+2205$; $x^2-106x=-220$

3. $9(7+2x^2)=25.(4x-3)$; $63+18x^2=100x-75$;
$18x^2-100x=-138$, etc.

4. $6x-11=249-2x^2$; $2x^2+6x=260$; $x^2+3x=130$, etc.

5. $6-x=4-4\sqrt{(2x-1)}+2x-1$; $4\sqrt{(2x-1)}=3x-3$;
$16(2x-1)=9x^2-18x+9$; $9x^2-50x=-25$, etc.

6. $x+12=2\sqrt{(4-3x)}$; $x^2+24x+144=4(4-3x)$; $x^2+36x=$

7. $2x+7+2\sqrt{(6x^2-15x-126)}+3x-18=7x+1$;
$\sqrt{(6x^2-15x-126)}=x+6$; $6x^2-15x-126=x^2+12x+$

$\text{ł}(204-5x)=400-40\sqrt{(3x-68)}+3x-68$;

$40\sqrt{(3x-68)}=23x-484$;

$1600(3x-68)=529x^2-22264x+234256$;

$529x^2-27064x=-343056$;

$$x^2-\frac{27064}{529}x+\frac{183115024}{(529)^2}=\frac{183115024-181476624}{(529)^2}=\frac{1638400}{(529)^2}$$

$$x-\frac{13532}{529}=\pm\frac{1280}{529}\text{, etc.}$$

$\sqrt{x}-4)(\sqrt{x}+4)=33$; $x-16=33$; $x=49$.

$\sqrt{x}+11).(\sqrt{x}-11)=608$; $x-121=608$; $x=729$.

$\sqrt{(x^2+17x+60)}=12$; $x^2+17x+60=144$; $x^2+17x=-84$, etc.

$:+3+2\sqrt{(x^2+11x+24)}+x+8=25x$; $2\sqrt{(x^2+11x+24)}=23x-11$;

$4(x^2+11x+24)=529x^2-506x+121$; $525x^2-550x=-25$;

$$x^2-\frac{22x}{21}=-\frac{1}{21}\text{, etc.}$$

$\sqrt{(25+x)}=8-\sqrt{(25-x)}$; $25+x=64-16\sqrt{(25-x)}+25-x$;

$8\sqrt{(25-x)}=32-x$; $64(25-x)=1024-64x+x^2$; $x^2=576$;

$x=\pm 24$.

$x+4+2\sqrt{(2x^2+7x-4)}+2x-1=36$; $2\sqrt{(2x^2+7x-4)}=33-3x$,

$4(2x^2+7x-4)=1089-198x+9x^2$; $x^2-226x=-1105$, etc.

$\sqrt{(13x-1)}=5+\sqrt{(2x-1)}$; $13x-1=25+10\sqrt{(2x-1)}+2x-1$;

$11x-25=10\sqrt{(2x-1)}$; $121x^2-550x+625=100(2x-1)$;

$$121x^2-750x=-725;\ x^2-\frac{750}{121}x+\frac{140625}{(121)^2}=\frac{140625-87725}{(121)^2}\text{, etc.}$$

$7x+1-2\sqrt{(21x^2+10x+1)}+3x+1=4$; $\sqrt{(21x^2+10x+1)}=5x-1$;

$21x^2+10x+1=25x^2-10x+1$; $4x^2=20x$; $x=5$ or 0.

$\sqrt{(4+x)}=3-\sqrt{x}$; $4+x=9-6\sqrt{x}+x$; $\sqrt{x}=\frac{5}{6}$; $x=\frac{25}{36}$.

$x+\sqrt{(x^2+9975)}=525$; $\sqrt{(x^2+9975)}=525-x$;

$x^2+9975x=275625-1050x+x^2$; $11025x=275625$, etc.

19. $\frac{x}{4}+3+2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{2x}{3}$; $2\sqrt{\left(\frac{x^2}{16}-9\right)}=\frac{x}{6}$;

$\frac{x^2}{16}-9=\frac{x^2}{144}$; $\frac{8x^2}{144}=9$; $x^2=9\times 18$; $x=\pm 9\sqrt{2}$.

20. $x^2-1+6\sqrt{(x^2-1)}=16$; $x^2-1+6\sqrt{(x^2-1)}+9=25$;
$\sqrt{(x^2-1)}+3=\pm 5$; $\sqrt{(x^2-1)}=2$ or -8.
Hence $x^2-1=4$ or 64; $x^2=5$ or 65, etc.

21. $x^2-2ax+a^2+2ab+b^2=x^2+a^2+b^2-2ax+2bx-2ab$; $4ab=2bx$;
$x=2a$.

22. $x^2+2ax+a^2+2ab+b^2=b^2+a^2+x^2-2ab-2bx+2ax$; $4ab=-2bx$;
$x=-2a$.

23. $x+4-2\sqrt{(x^2+4x)}+x=x+\frac{3}{2}$; $x+\frac{5}{2}=2\sqrt{(x^2+4x)}$;

$x^2+5x+\frac{25}{4}=4x^2+16x$; $3x^2+11x=\frac{25}{4}$; $x^2+\frac{11x}{3}+\frac{121}{36}=\frac{196}{36}$;

$x+\frac{11}{6}=\pm\frac{14}{6}$; $x=\frac{1}{2}$ or $-\frac{25}{6}$.

24. $\sqrt{x}+1=x+\frac{5}{4}$; $\sqrt{x}=x+\frac{1}{4}$; $x-\sqrt{x}=-\frac{1}{4}$; $x-\sqrt{x}+\frac{1}{4}=0$;

$\sqrt{x}-\frac{1}{2}=0$; $x=\frac{1}{4}$.

25. $\sqrt{(4+x)}=\sqrt{3}+\sqrt{x}$; $4+x=3+2\sqrt{(3x)}+x$; $2\sqrt{(3x)}=1$;
$12x=1$, etc.

26. $\sqrt{(x+4)}=9-\sqrt{(x+5)}$; $x+4=81-18\sqrt{(x+5)}+x+5$;
$18\sqrt{(x+5)}=82$; $9\sqrt{(x+5)}=41$; $81(x+5)=1681$, etc.

27. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-24x+x^2$;
$20x=144$, etc.

28. $x^2-21=\sqrt{(x^2-9)}$; $x^4-42x^2+441=x^2-9$; $x^4-43x^2=-450$;

$x^4-43x^2+\frac{1849}{4}=\frac{49}{4}$; $x^2-\frac{43}{2}=\pm\frac{7}{2}$; $x^2=25$ or 18, etc.

$\sqrt{(50+x)}=2+\sqrt{(50-x)}$; $50+x=4+4\sqrt{(50-x)}+50-x$;
$x-2=2\sqrt{(50-x)}$; $x^2-4x+4=4(50-x)$; $x^2=196$, etc.

$\sqrt{(2x+4)}-1=\sqrt{\left(\frac{x}{2}+6\right)}$; $2x+4-2\sqrt{(2x+4)}+1=\frac{x}{2}+6$;
$\frac{3x}{2}-1=2\sqrt{(2x+4)}$; $9x^2-12x+4=16(2x+4)$; $9x^2-44x=60$;
$x^2-\frac{44}{9}x+\frac{484}{81}=\frac{1024}{81}$, etc.

$3+x+\sqrt{(3x+x^2)}=6$; $\sqrt{(3x+x^2)}=3-x$; $3x+x^2=9-6x+x^2$;
$9x=9$; $x=1$.

$\sqrt{(x-1)}+\sqrt{(x+1)}=1$; $\sqrt{(x-1)}=1-\sqrt{(x+1)}$;
$x-1=1-2\sqrt{(x+1)}+x+1$; $2\sqrt{(x+1)}=3$; $4(x+1)=9$, etc.

$3x+\sqrt{(4x-x^2)}=6x-2\sqrt{(4x-x^2)}$; $3\sqrt{(4x-x^2)}=3x$; $4x-x^2=x^2$;
$2x^2=4x$, etc.

$\sqrt{\{a-\sqrt{(ax+x^2)}\}}=\sqrt{a}-\sqrt{x}$; $a-\sqrt{(ax+x^2)}=a-2\sqrt{(ax)}+x$;
$\sqrt{(ax+x^2)}=2\sqrt{(ax)}-x$; $ax+x^2=4ax-4x\sqrt{(ax)}+x^2$;
$4x\sqrt{(ax)}=3ax$; whence $x=0$, or $4\sqrt{(ax)}=3a$; $16ax=9a^2$;
$16x=9a$, etc.

CXXIII.

1. $x-2=0$, or, $x-5=0$; $\therefore x=2$ or 5.

2. $x-3=0$, or, $x+7=0$; $\therefore x=3$ or -7.

3. $x+9=0$, or, $x+2=0$; $\therefore x=-9$ or -2.

4. $x-5a=0$, or, $x-6b=0$; $\therefore x=5a$ or $6b$.

5. $2x+7=0$, or, $3x-5=0$; $\therefore x=-\frac{7}{2}$ or $\frac{5}{3}$.

6. $19x-227=0$, or, $14x+83=0$; $\therefore x=\frac{227}{19}$ or $-\frac{83}{14}$.

7. $5x-4m=0$, or, $6x-11n=0$; $\therefore x=\frac{4m}{5}$ or $\frac{11n}{6}$.

8. $x^2+5ax+6a^2=0$, or, $x^2-7ax+12a^2=0$;
 $(x+2a)(x+3a)=0$, or, $(x-4a)(x-3a)=0$;
 $\therefore x+2a=0$, or, $x+3a=0$, or, $x-4a=0$, or, $x-3a=0$, etc.

9. $(x+2)(x-2)(x-a)(x-a)=0$; $\therefore x=\pm 2$ or a.

10. $x.x.(x-5)=0$, $\therefore x=0$ or 5.

11. $acx-2a+b=0$, or, $bcx+3a-b=0$;
 $\therefore acx=2a-b$, or, $bcx=b-3a$, etc.

12. $cx-d=0$, or, $cx-e=0$, etc.

CXXIV.

1. Let m be one of the roots of the first equation, and $\frac{1}{m}$ one of the roots of the second equation.
 Then $am^2+bm+c=0$, and $c'm^2+b'm+a'=0$.
 Multiply the first by a' and the second by c; then
 $aa'm^2+a'bm+a'c=0$, and $cc'm^2+b'cm+a'c=0$.
 Subtracting $(aa'-cc')m^2+(a'b-b'c)m=0$.
 Whence $(aa'-cc')m=-(a'b-b'c)$ . . . . (1).
 Again, multiply the first by c' and the second by a; then
 $ac'm^2+bc'm+cc'=0$, and $ac'm^2+ab'm+aa'=0$.
 Subtracting $(bc'-ab')m=cc'-aa'$;
 or $aa'-cc'=-(ab'-bc')m$ . . . . (2).
 Multiplying (1) and (2) together, we get
 $(aa'-cc')^2=(ab'-bc')(a'b-b'c)$.

2. $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$;
 $\therefore \alpha^2+2\alpha\beta+\beta^2=\frac{b^2}{a^2}$, and $2\alpha\beta=\frac{2c}{a}$
 $\therefore \alpha^2+\beta^2=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$.

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

$$\therefore\ ac\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=ac\left(x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1\right)$$
$$=ac\left(x^2-\frac{b^2-2ac}{ac}x+1\right)$$
$$=acx^2+(2ac-b^2)x+ac.$$

The roots are $\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$, and that these may be *equal*, we must have $b^2-4ac=0$, or $c=\dfrac{b^2}{4a}$.

Putting this for c in the expression ax^2+bx+c,

$$ax^2+bx+\frac{b^2}{4a}=\frac{4a^2x^2+4abx+b^2}{4a}=\left(\frac{2ax+b}{2\sqrt{a}}\right)^2.$$

$\alpha+\beta=1+a$, and $\alpha\beta=\dfrac{1+a+a^2}{2}$.

$\alpha^2+2\alpha\beta+\beta^2=1+2a+a^2$, and $2\alpha\beta=1+a+a^2$

$\therefore\ \alpha^2+\beta^2=a.$

CXXV.

$c-5)(x-6)=0$, or, $x^2-11x+30=0$.

$c-4)(x+5)=0$, or, $x^2+x-20=0$.

$c+2)(x+7)=0$, or, $x^2+9x+14=0$.

$x-\dfrac{1}{2}\Big)\left(x-\dfrac{2}{3}\right)=0$, or, $(2x-1)(3x-2)=0$, or, $6x^2-7x+2=0$.

$c-7)\left(x+\dfrac{5}{9}\right)=0$, or, $(x-7)(9x+5)=0$, or, $9x^2-58x-35=0$.

$c-\sqrt{3})(x+\sqrt{3})=0$, or, $x^2-3=0$.

$c-m-n)(x-m+n)=0$, or, $x^2-2mx+m^2-n^2=0$.

8. $\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)=0$, or, $x^2-\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0$.

9. $\left(x+\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$, or, $x^2+\frac{\alpha^2-\beta^2}{\alpha\beta}x-1=0$.

CXXVI.

1. One root is found by trial to be 2; then
$x^3-11x^2+36x-36=(x-2)(x^2-9x+18)=(x-2)(x-3)(x-6)$.

2. One root is found by trial to be 1; then
$x^3-7x^2+14x-8=(x-1)(x^2-6x+8)=(x-1)(x-2)(x-4)$.

3. One root is found by trial to be -1; then
$x^3-5x^2-46x-40=(x+1)(x^2-6x-40)=(x+1)(x-10)(x-4)$

4. One root is found by trial to be -1; then
$4x^3+6x^2+x-1=(x+1)(4x^2+2x-1)=(x+1)4.\left(x^2+\frac{x}{2}-\frac{1}{4}\right)$
and since the roots of $x^2+\frac{x}{2}-\frac{1}{4}=0$ are $\frac{\sqrt{5}-1}{4}$ and $\frac{-\sqrt{5}-1}{4}$
$x^2+\frac{x}{2}-\frac{1}{4}=\left(x+\frac{1-\sqrt{5}}{4}\right).\left(x+\frac{1+\sqrt{5}}{4}\right)$, and therefore
$4x^3+6x^2+x-1=4(x+1)\left(x+\frac{1-\sqrt{5}}{4}\right)\left(x+\frac{1+\sqrt{5}}{4}\right)$.

5. One root is found by trial to be -1; then
$6x^3+11x^2-9x-14=(x+1)(6x^2+5x-14)=(x+1)(x+2)(6x-7)$.

6. If we put $x=-y-z$, the expression becomes zero, which shows that $x+y+z$ is one of its factors; the other is found by division to be $x^2+y^2+z^2-xy-xz-yz$.

7. If we put $a=b+c$, the expression becomes zero, which shows that $a-b-c$ is one of its factors; the other is found by division to be $a^2+b^2+c^2+ab+ac-bc$.

3. One root is found by trial to be 1; then

$$3x^3 - x^2 - 23x + 21 = (x-1)(3x^2 + 2x - 21) = (x-1)(3x-7)(x+3).$$

One root is found by trial to be 1; then

$$2x^3 - 5x^2 - 17x + 20 = (x-1)(2x^2 - 3x - 20) = (x-1)(x-4)(2x+5).$$

One root is found by trial to be -1; then

$$15x^3 + 41x^2 + 5x - 21 = (x+1)(15x^2 + 26x - 21)$$
$$= (x+1)(3x+7)(5x-3).$$

CXXVII.

$x^4 - 12x^2 = 13$; $x^4 - 12x^2 + 36 = 49$; $x^2 - 6 = \pm 7$, etc.

$x^6 + 14x^3 = -24$; $x^6 + 14x^3 + 49 = 25$; $x^3 + 7 = \pm 5$, etc.

$x^8 + 22x^4 = -21$; $x^8 + 22x^4 + 121 = 100$; $x^4 + 11 = \pm 10$, etc.

$x^{2m} + 3x^m + \frac{9}{4} = \frac{25}{4}$; $x^m + \frac{3}{2} = \pm\frac{5}{2}$, etc.

$x^{4n} - \frac{5}{3}x^{2n} + \frac{25}{36} = \frac{100}{36}$; $x^{2n} - \frac{5}{6} = \pm\frac{10}{6}$, etc.

6. $x - \frac{9}{2}x^{\frac{1}{2}} + \frac{81}{16} = \frac{121}{16}$; $x^{\frac{1}{2}} - \frac{9}{4} = \pm\frac{11}{4}$, etc.

7. $x^{-2} + 3x^{-1} + \frac{9}{4} = \frac{97}{36}$; $x^{-1} + \frac{3}{2} = \pm\frac{\sqrt{97}}{6}$; $x^{-1} = \frac{-9 \pm \sqrt{97}}{6}$

$\therefore x = \frac{6}{-9 \pm \sqrt{97}}$.

8. $x^{-2n} - x^{-n} + \frac{1}{4} = \frac{81}{4}$; $x^{-n} - \frac{1}{2} = \pm\frac{9}{2}$; $x^{-n} = 5$ or -4,

$\therefore \frac{1}{x^n} = \frac{1}{5}$ or $-\frac{1}{4}$; $x = \left(\frac{1}{5}\right)^{\frac{1}{n}}$ or $\left(-\frac{1}{4}\right)^{\frac{1}{n}}$.

9. $x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$;

$(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} + 9 = 25$; $(x^2 - 2x + 5)^{\frac{1}{2}} + 3 = \pm 5$;

$(x^2 - 2x + 5)^{\frac{1}{2}} = 2$ or -8; $x^2 - 2x + 5 = 4$ or 64, etc.

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;

$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39$;

$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=64$;

$\sqrt{(2x^2-5x+6)}+5=\pm 8$; $\sqrt{(2x^2-5x+6)}=3$ or -13;

$2x^2-5x+6=9$ or 169, etc.

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;

$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8$;

$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=a^2+2a+1$;

$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1)$;

$3x^2-2ax+4=(4+a)^2$ or $(2-a)^2$, etc.

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;

$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-2a+1$;

$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1)$; $x^2-ax+a^2=a^2$ or $(2-a)^2$, etc.

CXXVIII.

1. $\dfrac{2}{3}, \dfrac{6}{7}, \dfrac{7}{9}$; $\dfrac{42}{63}, \dfrac{54}{63}, \dfrac{49}{63}$, etc.

2. $\dfrac{x+3y}{x+2y}, \dfrac{x+2y}{x+y}$; $\dfrac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \dfrac{x^2+4xy+4y^2}{(x+2y)(x+y)}$, etc.

3. $\dfrac{x-5y}{x-4y}, \dfrac{x-3y}{x-2y}$; $\dfrac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \dfrac{x^2-7xy+12y^2}{(x-4y)(x-2y)}$, etc.

4. Let x be the number.

Then $\dfrac{a+x}{b+x}=\dfrac{c}{d}$; $ad+dx=bc+cx$;

$(c-d)x=ad-bc$; $x=\dfrac{ad-bc}{c-d}$.

5. Let $x:y$ be the ratio. Then

$$\left.\begin{aligned} x^2+y^2&=181 \\ xy&=90 \end{aligned}\right\}, \quad \left.\begin{aligned} x^2+y^2+2xy&=361 \\ x^2+y^2-2xy&=1 \end{aligned}\right\}, \quad \left.\begin{aligned} x+y&=19 \\ x-y&=\pm 1 \end{aligned}\right\}, \quad \begin{aligned} x&=10 \text{ or } 9 \\ y&=9 \text{ or } 10 \end{aligned}$$

CXXIX.

$x:y=6:9=2:3.$ 2. $x:y=b:a.$

$(a-c)x=(b+d)y$; $x:y=b+d:a-c.$

$x^2+2xy+y^2=6y^2$; $x+y=\pm\sqrt{6}.y$;

$x=(\pm\sqrt{6}-1)y$; $x:y=\pm\sqrt{6}-1:1.$

$x^2-12xy+36y^2=49y^2$; $x-6y=\pm 7y$;

$x=13y$ or $-y$; $x:y=13:1$, or, $-1:1.$

$x^2+mxy+\dfrac{m^2y^2}{4}=\dfrac{(m^2+4n^2)y^2}{4}$; $x+\dfrac{my}{2}=\dfrac{\pm\sqrt{(m^2+4n^2)}y}{2}$;

$x=\dfrac{-m\pm\sqrt{(m^2+4n^2)}}{2}y$, etc.

Let $3x$ and $4x$ be the numbers. Then

$\dfrac{3x+4x}{9x^2+16x^2}=\dfrac{7}{50}$; $7\times 25x^2=50\times 7x$; $x=2.$

Hence the numbers are 6 and 8.

Let $6x$ and $7x$ be the numbers. Then

$\dfrac{6x+12}{7x+12}=\dfrac{12}{13}$; $78x+156=84x+144$; $6x=12$; $x=2$, etc.

Let $7x$ and $13x$ be the numbers. Then $20x=100$; $x=5$, etc.

Let x and y be the numbers. Then $x^2-y^2=48$, and $\dfrac{x+y}{x-y}=\dfrac{12}{1}.$

Hence $x+y=12x-12y$, or, $13y=11x.$

Then $x^2-\dfrac{121x^2}{169}=48$; $48x^2=169\times 48$; $x=13$, etc.

Let x be the value of a gold coin, and y the value of a silver one.

Then $5x+4y=3x+12y$; $2x=8y$; $x:y=4:1.$

Let x be the value of a gold coin, and y the value of a silver one.

Then $8x+9y=6x+19y$; $2x=10y$; $y:x=1:5.$

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;

$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39$;

$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=6$

$\sqrt{(2x^2-5x+6)}+5=\pm 8$; $\sqrt{(2x^2-5x+6}$

$2x^2-5x+6=9$ or 169, etc.

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;

$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2$

$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=$

$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1)$;

$3x^2-2ax+4=(4+a)^2$ or $(2-a)^2$, etc.

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;

$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-$

$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1)$; x^2-ax+

CXXVIII.

1. $\frac{2}{3}, \frac{6}{7}, \frac{7}{9}; \frac{42}{63}, \frac{54}{63}, \frac{49}{63}$, etc.

2. $\frac{x+3y}{x+2y}, \frac{x+2y}{x+y}; \frac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \frac{x^2+4xy}{(x+2y)(}$

3. $\frac{x-5y}{x-4y}, \frac{x-3y}{x-2y}; \frac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \frac{x^2-7x}{(x-4}$

4. Let x be the number.

Then $\frac{a+x}{b+x}=\frac{c}{d}$; $ad+dx=bc+cx$

$(c-d)x=ad-bc$; $x=\frac{ad}{}$

5. Let $x:y$ be the

$x^2+y^2=181$

$xy=$

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;

$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39$;

$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=64$;

$\sqrt{(2x^2-5x+6)}+5=\pm 8$; $\sqrt{(2x^2-5x+6)}=3$ or -13;

$2x^2-5x+6=9$ or 169, etc.

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;

$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8$;

$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=a^2+2a+1$;

$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1)$;

$3x^2-2ax+4=(4+a)^2$ or $(2-a)^2$, etc.

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;

$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-2a+1$;

$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1)$; $x^2-ax+a^2=a^2$ or $(2-a)^2$, etc.

CXXVIII.

1. $\frac{2}{3}, \frac{6}{7}, \frac{7}{9}; \frac{42}{63}, \frac{54}{63}, \frac{49}{63}$, etc.

2. $\frac{x+3y}{x+2y}, \frac{x+2y}{x+y}; \frac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \frac{x^2+4xy+4y^2}{(x+2y)(x+y)}$, etc.

3. $\frac{x-5y}{x-4y}, \frac{x-3y}{x-2y}; \frac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \frac{x^2-7xy+12y^2}{(x-4y)(x-2y)}$, etc.

4. Let x be the number.

Then $\frac{a+x}{b+x}=\frac{c}{d}$; $ad+dx=bc+cx$;

$(c-d)x=ad-bc$; $x=\frac{ad-bc}{c-d}$.

5. Let $x:y$ be the ratio. Then

$$\left.\begin{array}{r}x^2+y^2=181\\ xy\ =\ 90\end{array}\right\}, \quad \left.\begin{array}{r}x^2+y^2+2xy=361\\ x^2+y^2-2xy=\quad 1\end{array}\right\}, \quad \left.\begin{array}{r}x+y=19\\ x-y=\pm 1\end{array}\right\}, \quad \begin{array}{l}x=10 \text{ or } 9\\ y=\ 9 \text{ or } 10.\end{array}$$

CXXIX.

$x:y=6:9=2:3.$ 2. $x:y=b:a.$

$(a-c)x=(b+d)y$; $x:y=b+d:a-c.$

$x^2+2xy+y^2=6y^2$; $x+y=\pm\sqrt{6}.y$;

$x=(\pm\sqrt{6}-1)y$; $x:y=\pm\sqrt{6}-1:1.$

$x^2-12xy+36y^2=49y^2$; $x-6y=\pm 7y$;

$x=13y$ or $-y$; $x:y=13:1$, or, $-1:1$.

$$x^2+mxy+\frac{m^2y^2}{4}=\frac{(m^2+4n^2)y^2}{4};\ x+\frac{my}{2}=\frac{\pm\sqrt{(m^2+4n^2)}y}{2};$$

$$x=\frac{-m\pm\sqrt{(m^2+4n^2)}}{2}y, \text{ etc.}$$

Let $3x$ and $4x$ be the numbers. Then

$$\frac{3x+4x}{9x^2+16x^2}=\frac{7}{50};\ 7\times 25x^2=50\times 7x;\ x=2.$$

Hence the numbers are 6 and 8.

Let $6x$ and $7x$ be the numbers. Then

$$\frac{6x+12}{7x+12}=\frac{12}{13};\ 78x+156=84x+144;\ 6x=12;\ x=2, \text{ etc.}$$

Let $7x$ and $13x$ be the numbers. Then $20x=100$; $x=5$, etc.

Let x and y be the numbers. Then $x^2-y^2=48$, and $\frac{x+y}{x-y}=\frac{12}{1}$.

Hence $x+y=12x-12y$, or, $13y=11x$.

Then $x^2-\frac{121x^2}{169}=48$; $48x^2=169\times 48$; $x=13$, etc.

Let x be the value of a gold coin, and y the value of a silver one.

Then $5x+4y=3x+12y$; $2x=8y$; $x:y=4:1$.

Let x be the value of a gold coin, and y the value of a silver one.

Then $8x+9y=6x+19y$; $2x=10y$; $y:x=1:5$.

CXXX.

1. $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$.

2. $\frac{3}{7}\times\frac{14}{9}\times\frac{4}{3}=\frac{8}{9}$.

3. $\frac{x^2-y^2}{x^3+y^3}\times\frac{x^2-xy+y^2}{x+y}=\frac{x-y}{x+y}$.

4. $\frac{a^2-b^2+2bc-c^2}{a^2-b^2-2bc-c^2}\times\frac{a+b+c}{a+b-c}=\frac{(a+b-c)(a-b+c)(a+b+c)}{(a+b+c)(a-b-c)(a+b-c)}$, etc.

5. $\frac{m^3+n^3}{m^3-n^3}\times\frac{m-n}{m+n}=\frac{m^2-mn+n^2}{m^2+mn+n^2}$.

6. $\frac{(x+2)(x+3)}{(y-3)(y-4)}\times\frac{y(y-3)}{x(x+3)}=\frac{(x+2)y}{(y-4)x}$.

CXXXI.

1. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$ and $c=\lambda d$.

Then $\frac{a+b}{a}=\frac{\lambda b+b}{\lambda b}=\frac{\lambda+1}{\lambda}$,

and $\frac{c+d}{c}=\frac{\lambda d+d}{\lambda d}=\frac{\lambda+1}{\lambda}$.

2. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$, $c=\lambda d$.

Then $\frac{a^2-b^2}{b^2}=\frac{\lambda^2b^2-b^2}{b^2}=\lambda^2-1$,

and $\frac{c^2-d^2}{d^2}=\frac{\lambda^2d^2-d^2}{d^2}=\lambda^2-1$.

3. Let $\frac{a_1}{b_1}=\lambda$, then $\frac{a_2}{b_2}=\lambda$. Then $a_1=\lambda b_1$, and $a_2=\lambda b_2$.

Then $\frac{m_1a_1+m_2a_2}{m_1b_1+m_2b_2}=\frac{m_1\lambda b_1+m_2\lambda b_2}{m_1b_1+m_2b_2}=\lambda=\frac{a_1}{b_1}$.

4. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{3a^2+ab+2b^2}{3a^2-2b^2} = \frac{3b^2\lambda^2+b^2\lambda+2b^2}{3b^2\lambda^2-2b^2} = \frac{3\lambda^2+\lambda+2}{3\lambda^2-2}$$

$$\frac{3c^2+cd+2d^2}{3c^2-2d^2} = \frac{3d^2\lambda^2+d^2\lambda+2d^2}{3d^2\lambda^2-2d^2} = \frac{3\lambda^2+\lambda+2}{3\lambda^2-2}.$$

5. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{a^2+3ab+b^2}{c^2+3cd+d^2} = \frac{b^2\lambda^2+3b^2\lambda+b^2}{d^2\lambda^2+3d^2\lambda+d^2} = \frac{b^2(\lambda^2+3\lambda+1)}{d^2(\lambda^2+3\lambda+1)} = \frac{b^2}{d^2}$$

$$\frac{2ab+3b^2}{2cd+3d^2} = \frac{2b^2\lambda+3b^2}{2d^2\lambda+3d^2} = \frac{b^2(2\lambda+3)}{d^2(2\lambda+3)} = \frac{b^2}{d^2}.$$

6. Let $\frac{a}{b} = \lambda$; then $\frac{c}{d} = \lambda$ and $\frac{e}{f} = \lambda$.

Then $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\dfrac{mc-ne}{md-nf} = \dfrac{md\lambda-nf\lambda}{md-nf} = \lambda = \dfrac{a}{b}$.

7. $$\frac{a-\frac{ma}{n}}{b-\frac{mb}{n}} = \frac{a\left(1-\frac{m}{n}\right)}{b\left(1-\frac{m}{n}\right)} = \frac{a}{b}.$$

8. Let $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\dfrac{ac}{bd} = \dfrac{bd\lambda^2}{bd} = \lambda^2$

and $\dfrac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2} = \dfrac{lb^2\lambda^2+md^2\lambda^2+nf^2\lambda^2}{lb^2+md^2+nf^2} = \lambda^2.$

9. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_3\lambda$.

Then $\dfrac{a_1^2+a_2^2+a_3^2}{b_1^2+b_2^2+b_3^2} = \dfrac{b_1^2\lambda^2+b_2^2\lambda^2+b_3^2\lambda^2}{b_1^2+b_2^2+b_3^2} = \lambda^2 = \dfrac{a_1^2}{b_1^2}.$

10. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_3\lambda$.

Then $\dfrac{a_1a_2+a_2a_3+a_3a_1}{b_1b_2+b_2b_3+b_3b_1} = \dfrac{b_1b_2\lambda^2+b_2b_3\lambda^2+b_3b_1\lambda^2}{b_1b_2+b_2b_3+b_3b_1} = \lambda^2 = \dfrac{a_1^2}{b_1^2}.$

11. $(a^2-ab+b^2)(c^2+cd+d^2)=(a^2+ab+b^2)(c^2-cd+d^2)$;
$(a^2+b^2)(c^2+cd+d^2-c^2+cd-d^2)=ab(c^2-cd+d^2+c^2+cd+d^2)$;
$(a^2+b^2)(2cd)=ab(2c^2+2d^2)$; $a^2cd+b^2cd=abc^2+abd^2$;
$a^2cd-abd^2=abc^2-b^2cd$; $ad(ac-bd)=bc(ac-bd)$;
then *either* $ad=bc$; and therefore $a:b=c:d$,
or $ac-bd=0$; and therefore $ac=bd$, or, $a:b=d:c$.

12. $(a^2+b^2)(c^2-d^2)=(a^2-b^2)(c^2+d^2)$;
$a^2c^2+b^2c^2-a^2d^2-b^2d^2=a^2c^2+a^2d^2-b^2c^2-b^2d^2$; $2b^2c^2=2a^2d^2$;
$bc=ad$; and $\therefore a:b=c:d$.

13. Let $\frac{a}{b}=\lambda$ and $\frac{c}{d}=\lambda$; then $a=b\lambda$ and $c=d\lambda$.

Then $\frac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\frac{(b\lambda+d\lambda)(b^2\lambda^2+d^2\lambda^2)}{(b\lambda-d\lambda)(b^2\lambda^2-d^2\lambda^2)}=\frac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}$.

14. Let $a_1=b_1\lambda$ and $a_2=b_2\lambda$.

Then $\frac{\sqrt{(a_1^2+a_2^2)}}{\sqrt{(b_1^2+b_2^2)}}=\sqrt{\frac{b_1^2\lambda^2+b_2^2\lambda^2}{b_1^2+b_2^2}}=\sqrt{(\lambda^2)}=\lambda=\frac{a_1}{b}$.

CXXXII.

1. $(a-b)c=(b-c)b$; $ac-bc=b^2-bc$; $ac=b^2$; $a:b=b:c$.

2. Let $a=\lambda b$ and $c=\lambda d$.

Then $\frac{(a^2+b^2)(a+b)}{a^3}=\frac{(\lambda^2b^2+b^2)(\lambda b+b)}{\lambda^3b^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$,

and $\frac{(c^2+d^2)(c+d)}{c^3}=\frac{(\lambda^2d^2+d^2)(\lambda d+d)}{\lambda^3d^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$.

Also, $\frac{\sqrt[4]{(ma^4+nc^4)}}{\sqrt[4]{(mb^4+nd^4)}}=\sqrt[4]{\frac{mb^4\lambda^4+nd^4\lambda^4}{mb^4+nd^4}}=\sqrt[4]{\lambda^4}=\lambda=\frac{a}{b}$.

3. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{ma-nb}{ma+nb}=\frac{mb\lambda-nb}{mb\lambda+nb}=\frac{m\lambda-n}{m\lambda+n}$,

and $\frac{mc-nd}{mc+nd}=\frac{md\lambda-nd}{md\lambda+nd}=\frac{m\lambda-n}{m\lambda+n}$.

4. $(5a+3b)(7b+3c)=(7a+3b)(5b+3c)$;
$35ab+15ac+21b^2+9bc=35ab+21ac+15b^2+9bc$; $6b^2=6ac$;
$b^2=ac$; $a:b=b:c$.

5. Let $a:b=c:d$, and suppose a to be the greatest of the four.

Then $\because \frac{a}{b}=\frac{c}{d}$, and a is greater than b, c is greater than d,

and $\because \frac{a}{c}=\frac{b}{d}$, and a is greater than c, b is greater than d.

Again, if $a+b:m+n=m-n:a-b$,
and $a+b$ is greater than $m+n$
$m-n$ is greater than $a-b$
$\therefore a+b+m-n$ is greater than $m+n+a-b$
$\therefore 2b$ is greater than $2n$; and $\therefore b$ is greater than n.

6. $(x-1)(x+2)=(x-2)(2x+1)$; $x^2+x-2=2x^2-3x-2$; $x^2=4x$;
$x=4$ or 0.

7. $\frac{a}{b}+1=\frac{c}{d}+1$; $\frac{a}{b}=\frac{c}{d}$.

8. Suppose the bicycle went $5x$ and the tricycle $4x$ yards per minute.
Then the tricycle had a start of $2x$ yards out of 1760 yards.
Also the bicycle went 880 yards while the tricycle went
$1760-2x-176$. $\therefore 1760:1760-2x-176=5:4$;
$1760\times 4=(1584-2x)\times 5$; $10x=7920-7040$, etc.

9. $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$; $\therefore \frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$;

$\therefore \frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$; and since b^2 is greater than d^2,
$\therefore a^2-b^2$ is greater than c^2-d^2
$\therefore a^2+d^2$ is greater than b^2+c^2.

10. $(10a+b)(12c+d)=(10c+d)(12a+b)$;
$120ac+12bc+10ad+bd=120ac+12ad+10bc+bd$; $2bc=2ad$
$bc=ad$; $a:b=c:d$.

11. $2x=3y$, and $xy=600$.

Hence $x\times\frac{2x}{3}=600$; $x^2=900$; $x=30$, etc.

12. Let $\frac{a}{b}=\lambda$; then $\frac{b}{c}=\lambda$.

Then (1.) $\frac{a}{a+b}=\frac{\lambda b}{\lambda b+b}=\frac{\lambda}{\lambda+1}$

$$\frac{a-b}{a-c}=\frac{\lambda b-b}{\lambda b-c}=\frac{\lambda^2c-\lambda c}{\lambda^2c-c}=\frac{\lambda^2-\lambda}{\lambda^2-1}=\frac{\lambda}{\lambda+1}.$$

(2.) $(a^2+b^2)(b^2+c^2)=(\lambda^2b^2+b^2)(\lambda^2c^2+c^2)$

$=\lambda^2c^2(\lambda^2+1)(\lambda^2c^2+c^2)=\lambda^2c^4(\lambda^2+1)^2$

$(ab+bc)^2=(\lambda b^2+\lambda c^2)^2=(\lambda^3c^2+\lambda c^2)^2=\lambda^2c^4(\lambda^2+1)^2$.

13. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{a+b}{b}=\frac{b\lambda+b}{b}=\lambda+1$;

and $\frac{c+d}{d}=\frac{d\lambda+d}{d}=\lambda+1$.

Hence $\frac{ab-bc-dx+(bc+dx)}{bc+dx}=\frac{a-b-c+(b+c)}{b+c}$;

or, $\frac{ab}{bc+dx}=\frac{a}{b+c}$; $\frac{b}{bc+dx}=\frac{1}{b+c}$; $b^2+bc=bc+dx$, etc.

14. Let $a=\lambda b$, and $b=\lambda c$.

Then $\frac{a+mb}{a-mb}=\frac{\lambda b+mb}{\lambda b-mb}=\frac{\lambda+m}{\lambda-m}$;

and $\frac{b+mc}{b-mc}=\frac{\lambda c+mc}{\lambda c-mc}=\frac{\lambda+m}{\lambda-m}$.

15. $a=\frac{5b}{4}$; and therefore $\frac{a^2-b^2}{a^2+b^2}=\frac{\frac{25b^2}{16}-b^2}{\frac{25b^2}{16}+b^2}=\frac{9}{41}$.

16. Let the sides be $2\frac{1}{2}x$, $3\frac{3}{4}x$, and $4x$ yards.

Then $2\frac{1}{2}x+3\frac{3}{4}x+4x=205$; $\frac{41}{4}x=205$; $x=20$, etc.

7. Let the sides be $3x$, $4x$ and $5x$ yards.
Then $3x+4x+5x=480$; $12x=480$; $x=40$, etc.

8. Let $a+b$ be the greatest term of the proportion, then $a-b$ is the least.
Also, $\frac{a+b}{p+q}=\frac{p-q}{a-b}$.
Hence $\frac{(a+b)-(p+q)}{p+q}=\frac{(p-q)-(a-b)}{a-b}$.
Now $p+q$ is greater than $a-b$.
$\therefore$ $(a+b)-(p+q)$ is greater than $(p-q)-(a-b)$.
And $\therefore$ $(a+b)+(a-b)$ is greater than $(p+q)+(p-q)$.

9. Let x be the rate of the man's rowing, y the rate of the stream, in miles per hour.
Then $x+y=$ the man's rate down stream,
and $x-y=$ the man's rate up stream.
Hence $x+y:x-y=5:3$; $3x+3y=5x-5y$; $4y=x$.
Also $\frac{30}{x+y}+\frac{30}{x-y}=12$; $\frac{30}{5y}+\frac{30}{3y}=12$; $16=12y$; $y=1\frac{1}{3}$.

10. Let C contain x pints of brandy and y pints of water.
Then A contains $x+y$ pints of water, and B $x+y$ pints of brandy.
Hence when B and C are mixed, the mixture contains $2x+y$ pints of brandy;
and when A and C are mixed, the mixture contains x pints of brandy;
therefore $2x+y:x=9:1$; $2x+y=9x$; $y=7x$, etc.

11. Let x be the number of quarters ; y the price of each in shillings.
Then selling price : $xy=105:100$
$\therefore$ $\frac{105xy}{100}=xy+16\times 20$; $\therefore$ $xy=6400$.
Again $x(y+5)=xy+20y$; $\therefore$ $x=4y$.
Hence $4y^2=6400$; $y^2=1600$; $y=40$; $x=160$.

22. Let x be the price of the horse in pounds.

Then since cost price : gain $=100$: gain per cent.

$\therefore x : 144-x=100 : x$

$\therefore x^2+100x=14400$; $x^2+100x+2500=16900$; $x=80$.

23. Let x be the cost of the goods in pounds.

Then $x : 96-x=100 : x$; $x^2+100x=9600$; $x=60$.

24. Let x be the cost of the sheep in pounds.

Then $x : 24-x=100 : x$; $x^2+100x=2400$; $x=20$.

25. Let the first crew row x yards, and the second y yards at each stroke.

Then in 8 minutes the first crew row $320x$ yards,

and in 8 minutes the second crew row $360y$ yards.

Also the second crew has $4x$ yards start.

Hence $320x=360y+4x$; $316x=360y$; $79x=90y$, etc.

26. Let x be the rate of the fast train, y the rate of the slow train.

Then $\frac{180}{y}+\frac{30}{y}=$ time for journey by slow train

$\frac{180}{x}+\frac{15}{y}=$ time for journey by fast train

$\therefore \frac{210}{y} : \frac{180y+15x}{xy}=14:9$; or, $\frac{14}{y} : \frac{12y+x}{xy}=14:9$.

Hence $14\times 9\times x=(12y+x)\times 14$; $8x=12y$; $2x=3y$.

Again $x=y+15$; $\therefore 2y+30=3y$; $y=30$; $x=45$.

27. Let x be the worth of the article, y the selling price, in pounds.

Then $x : x-y=100 : x$

$\therefore x^2=100x-100y$; $x^2-100x+2500=2500-100y$.

To obtain a real value for x, $100y$ must not be greater than 2500;

$\therefore y$ cannot be greater than 25.

CXXXIII.

1. Let $A=\frac{m}{B}$ and $B=\frac{n}{C}$.

Then $A=m\div\frac{n}{C}=\frac{m}{n}\cdot C$; $\therefore A \propto C$.

2. Let $A=mB$. Then $\dfrac{A}{P}=m\cdot\dfrac{B}{P}$; $\therefore \dfrac{A}{P}\propto\dfrac{B}{P}$.

3. Let $A=mB$ and $C=nD$. Then $AC=mn.BD$, etc.

4. $5:7=12:x$; $5x=84$, etc.

5. $x:\dfrac{1}{y}=10:\dfrac{1}{2}$; $\dfrac{x}{2}=\dfrac{10}{y}$; $\dfrac{4}{2}=\dfrac{10}{y}$; $y=5$.

6. $x:yz=1:2\times 3$; $6x=yz$; $6\times 4=y\times 2$; $y=12$.

7. $x:\dfrac{y}{z}=6:\dfrac{4}{3}$; $\dfrac{4x}{3}=\dfrac{6y}{z}$; $\dfrac{4x}{3}=\dfrac{6\times 5}{7}$; $x=3\dfrac{3}{14}$.

8. $3x+5y:5x+3y=31:25$; $75x+125y=155x+93y$; $32y=80x$, etc.

9. Let $A=mB$, and $B^3=nC^2$; then $\dfrac{A^3}{m^3}=nC^2$; $A^3=\dfrac{n}{m^3}\cdot C^2$;

$A=\dfrac{n^{\frac{1}{3}}}{m}\cdot C^{\frac{2}{3}}$; $\therefore A\propto C^{\frac{2}{3}}$.

10. $z:xy=4:2$; $30:3x=2:1$; $x=5$.

11. $A:B=8:12$; $12A=8B$; $A=\dfrac{2}{3}B$.

12. $x^2:y^3=9:64$; $64x^2=9y^3$.

13. $x^2:\dfrac{1}{y^3}=4:\dfrac{1}{27}$; $\dfrac{x^2}{27}=\dfrac{4}{y^3}$; $x^2=\dfrac{108}{y^3}$.

14. $x^3:y^2=27:4$; $4x^3=27y^2$.

15. Let $x=mz$, and $y=\dfrac{n}{z}$; then $xy=mn$; $x=mn.\dfrac{1}{y}$.

16. The area of a triangle $=\frac{1}{2}$(base $\times$ altitude).
Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the triangles; then $a_1b_1=a_2b_2$, or, $a_1:a_2=b_2:b_1$.

17. The area of a parallelogram = base × altitude.
Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the parallelograms.
Then $a_1b_1 = a_2b_2$; or, $a_1 : a_2 = b_2 : b_1$.

18. Let $y = p + mx + nx^2$.

$$\left.\begin{aligned} \text{Then } 6 &= p + m + n \\ 11 &= p + 2m + 4n \\ 18 &= p + 3m + 9n \end{aligned}\right\};\quad \left.\begin{aligned} 5 &= m + 3n \\ 7 &= m + 5n \end{aligned}\right\};\ n = 1,\ m = 2,\ p = 3.$$

$\therefore\ y = 3 + 2x + x^2$.

19. $10 \times 27 : 9 \times 9 \times 10 = 2 \times 27 : 3 \times 3 \times$ required height in feet.

$$\text{Height} = \frac{9 \times 9 \times 10 \times 2 \times 27}{10 \times 27 \times 3 \times 3} \text{ ft.} = 18 \text{ ft.}$$

20. Let g, n, l, b represent the area of glass, the number, length, and breadth of the panes respectively.
Then suppose $g = m.nlb$, where m is a constant,

$$n = \frac{p}{b^2}, \text{ where } p \text{ is a constant.}$$

$$l = \frac{q}{b}, \text{ where } q \text{ is a constant.}$$

Then $$g = m.\frac{p}{b^2}.l.\frac{q}{l} = mpq.\frac{1}{b^2} = mpq.\frac{l^2}{q^2} = \frac{mp}{q}.l^2$$

$\therefore\ g$ varies as l^2.

CXXXIV.

1. $a = 2,\ d = 3,\ n - 1 = 16$

$$z = 2 + 16 \times 3 = 2 + 48 = 50.$$

2. $a = 4,\ d = 4,\ n - 1 = 49$

$$z = 4 + 4 \times 49 = 4 + 196 = 200.$$

3. $a = 7,\ d = \frac{1}{4},\ n - 1 = 15$

$$z = 7 + \frac{1}{4} \times 15 = 7 + \frac{15}{4} = 10\frac{3}{4}.$$

4. $a=\frac{1}{2}, d=-\frac{3}{2}, n-1=22$

$$s=\frac{1}{2}-\frac{3}{2}\times 22=\frac{1}{2}-33=-32\frac{1}{2}.$$

5. $a=\frac{5}{6}, d=\frac{1}{2}-\frac{5}{6}=-\frac{1}{3}, n-1=11$

$$s=\frac{5}{6}-\frac{1}{3}\times 11=\frac{5}{6}-\frac{11}{3}=-2\frac{5}{6}.$$

6. $a=-12, d=4, n-1=13$

$$s=-12+4\times 13=-12+52=40.$$

7. $a=-3, d=8, n-1=15$

$$s=-3+8\times 15=-3+120=117.$$

8. $a=\frac{n-1}{n}, d=\frac{n-2}{n}-\frac{n-1}{n}=\frac{-1}{n}, n-1=n-1$

$$s=\frac{n-1}{n}-\frac{1}{n}\times(n-1)=\frac{n-1}{n}-\frac{n-1}{n}=0.$$

9. $a=x^2+2xy+y^2, d=-2xy, n-1=n-1$

$$s=x^2+2xy+y^2-2xy(n-1)=x^2+y^2-2(n-2)xy.$$

10. $a=\frac{a-b}{a+b}, d=\frac{3a-2b}{a+b}, n-1=n-1$

$$s=\frac{a-b}{a+b}+\frac{3an-2bn-3a+2b}{a+b}=\frac{3an-2bn-2a+b}{a+b}.$$

CXXXV.

1. $2a=4, d=1, n=100$

$$s=\frac{100}{2}\{2+99\times 1\}=50\times 101=5050.$$

2. $2a=4, d=2, n=50$

$$s=25\{4+49\times 2\}=25\times 102=2550.$$

3. $2a=6,\ d=4,\ n=20$

$$s=10\{6+19\times 4\}=10\times 82=820.$$

4. $2a=\frac{1}{2},\ d=\frac{1}{4},\ n=15$

$$s=\frac{15}{2}\left\{\frac{1}{2}+\frac{1}{4}\times 14\right\}=\frac{15}{2}\times 4=30.$$

5. $2a=-18,\ d=2,\ n=12$

$$s=6\{-18+2\times 11\}=6\times 4=24.$$

6. $2a=\frac{5}{3},\ d=-\frac{1}{3},\ n=17$

$$s=\frac{17}{2}\left\{\frac{5}{3}-\frac{1}{3}\times 16\right\}=\frac{17}{2}\times\left(-\frac{11}{3}\right)=-\frac{187}{6}=-31\frac{1}{6}.$$

7. $2a=2,\ d=1,\ n=n$

$$s=\frac{n}{2}\{2+1\times(n-1)\}=\frac{n}{2}.(n+1).$$

8. $2a=2,\ d=3,\ n=n$

$$s=\frac{n}{2}\{2+3(n-1)\}=\frac{n}{2}(3n-1)=\frac{3n^2-n}{2}.$$

9. $2a=2,\ d=7,\ n=n$

$$s=\frac{n}{2}\{2+7(n-1)\}=\frac{n}{2}(7n-5)=\frac{7n^2-5n}{2}.$$

10. $2a=\frac{2(n-1)}{n},\ d=-\frac{1}{n},\ n=n$

$$s=\frac{n}{2}\left\{\frac{2n-2}{n}-\frac{n-1}{n}\right\}=\frac{n}{2}\times\frac{n-1}{n}=\frac{n-1}{2}.$$

CXXXVI.

1. $-14=100+19d$; $19d=-114$; $d=-6$.

2. $-x=x+50d$; $50d=-2x$; $d=-\frac{x}{25}$.

3. $5\frac{1}{2}=-\frac{1}{2}+48d$; $48d=6$; $d=\frac{1}{8}$.

4. $-21\frac{3}{4}=-\frac{3}{4}+24d$; $24d=-21$; $d=-\frac{7}{8}$.

5. $-20=-10+5d$; $5d=-10$; $d=-2$.

6. $0=150+90d$; $90d=-150$; $d=-\frac{5}{3}$.

CXXXVII.

1. (1.) $\left.\begin{array}{l} a+58d=70 \\ a+65d=84 \end{array}\right\}$; $7d=14$; $d=2$; $a=70-116=-46$.

(2.) $\left.\begin{array}{l} a+19d=93-35b \\ a+20d=98-37b \end{array}\right\}$; $d=5-2b$; $a+95-38b=93-35b$, etc.

(3.) $\left.\begin{array}{l} a+\quad d=\frac{1}{2} \\ a+54d=5{\cdot}8 \end{array}\right\}$; $\left.\begin{array}{r} 54a+54d=\ 27 \\ a+54d=5{\cdot}8 \end{array}\right\}$; $53a=21{\cdot}2$; $a=\frac{2}{5}$.

(4.) $\left.\begin{array}{l} a+\quad d=\quad 4 \\ a+86d=-30 \end{array}\right\}$; $\left.\begin{array}{r} 86a+86d=\ 344 \\ a+86d=-\ 30 \end{array}\right\}$; $85a=374$; $a=4{\cdot}4$.

2. $\left.\begin{array}{l} (a+2d)+(a+7d)=31 \\ (a+4d)+(a+9d)=43 \end{array}\right\}$; $\left.\begin{array}{l} 2a+\ 9d=31 \\ 2a+13d=43 \end{array}\right\}$; $d=3$, $a=2$.

Hence sum of 10 terms $=5\{4+27\}=155$.

3. $\left.\begin{array}{l} a+(a+2d)=0 \\ (a+d)+(a+6d=40 \end{array}\right\}$; $\left.\begin{array}{l} 2a+2d=\ 0 \\ 2a+7d=40 \end{array}\right\}$; $d=8$, $a=-8$.

Hence sum of 7 terms $=\frac{7}{2}\cdot\{-16+48\}=\frac{7}{2}\times 32=112$.

4. $\left.\begin{array}{l} a+3d=24 \\ a+4d=33 \end{array}\right\}$; $d=9$, $a=-3$.

Hence the 100th term is $-3+99\times 9=-3+891=888$.

5. $302 = 5 + (n-1) \times 3$; $300 = 3n$; $n = 100$.

6. $s = \frac{20}{2}\left\{32\frac{1}{6} + 19 \times 32\frac{1}{6}\right\} = 10 \times 20 \times \frac{193}{6} = 6433\frac{1}{3}$.

7. From the formula $s = (a+z)\frac{n}{2}$

$s = (1 + 103) \times 26 = 2704s. = £135.4s.$

8. (1.) 41st term $= -5 + 9 \times 40 = 355$

Sum $= (-5 + 355) \times \frac{41}{2} = 7175$.

(2.) 41st term $= 4a^2 - 4a^2 \times 40 = -156a^2$

Sum $= (4a^2 - 156a^2) \times \frac{41}{2} = -3116a^2$.

(3.) 41st term $= 1 + x + (4 + 2x)40 = 161 + 81x$

Sum $= (1 + x + 161 + 81x) \times \frac{41}{2} = 3321 + 1681x$.

(4.) 41st term $= -4\frac{1}{2} + 3\frac{1}{10} \times 40 = 119\frac{1}{2}$

Sum $= \left(-4\frac{1}{2} + 119\frac{1}{2}\right) \times \frac{41}{2} = 2357\frac{1}{2}$.

(5.) 41st term $= \frac{1}{4} + \frac{1}{5} \times 40 = 8\frac{1}{4}$

Sum $= \left(\frac{1}{4} + 8\frac{1}{4}\right) \times \frac{41}{2} = 174\frac{1}{4}$.

9. (1.) $a = 1002$; $d = -8$; nth term $= a + (n-1)d$

$2 = 1002 - 8(n-1)$; $8n = 1008$; $n = 126$

1662 $s = (1002 + 2) \times 63 = 63252$.

(2.) $186 = -6 + 8(n-1)$; $8n = 200$; $n = 25$

$s = (-6 + 186) \times \frac{25}{2} = 2250$.

(3.) $-72{\cdot}3x=2\frac{1}{2}x-1{\cdot}7x(n-1)$; $1{\cdot}7n=76{\cdot}5$; $n=45$

$s=(2{\cdot}5x-72{\cdot}3x)\times\frac{45}{2}=-1570{\cdot}5x.$

(4.) $-24=\frac{1}{2}-(n-1)\times\frac{1}{4}$; $\frac{n}{4}=24\frac{3}{4}$; $n=99$

$s=\left(\frac{1}{2}-24\right)\times\frac{99}{2}=-\frac{4653}{4}=-1163\frac{1}{4}.$

(5.) $139(1-m)=m-1+(n-1)2(1-m)$; $139=-1+2(n-1)$; $n=71$

$s=\{m-1+139(1-m)\}\times\frac{71}{2}=4899(1-m).$

(6.) $x-2=x+254-(n-1)\times 4$; $4n=2+258$; $n=65$

$s=\{x+254+x-2\}\frac{65}{2}=65x+8190.$

CXXXVIII.

$18=3+5d$; $d=3$; means are 6, 9, 12, 15.

$-2=2+6d$; $d=-\frac{2}{3}$; means are $1\frac{1}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-1\frac{1}{3}$.

$\frac{2}{3}=3+4d$; $d=-\frac{7}{12}$; means are $2\frac{5}{12}$, $1\frac{5}{6}$, $1\frac{1}{4}$.

$\frac{1}{3}=\frac{1}{2}+5d$; $d=-\frac{1}{30}$; means are $\frac{7}{15}$, $\frac{13}{30}$, $\frac{2}{5}$, $\frac{11}{30}$.

CXXXIX.

$n=m+4d$; $d=\frac{n-m}{4}$; means are $\frac{3m+n}{4}$, etc.

$m-1=m+1+5d$; $d=-\frac{2}{5}$; means are $\frac{5m+3}{5}$, etc.

3. $n^2+1=n^2+5d$; $d=\frac{1}{5}$; means are $\frac{5n^2+1}{5}$, etc.

4. $x^2-y^2=x^2+y^2+4d$; $d=-\frac{y^2}{2}$; means are $\frac{2x^2+y^2}{2}$, etc.

CXL.

1. $z=1\times 2^6=64$.

2. $z=4\times 3^9=4\times 19683=78732$.

3. $z=5\times 4^8=327680$.

4. $z=8\times\left(\frac{1}{2}\right)^{14}=\frac{2^3}{2^{14}}=\frac{1}{2^{11}}=\frac{1}{2048}$.

5. $z=2\times 3^8=13122$.

6. $z=\frac{1}{64}\times 4^{10}=\frac{4^{10}}{4^3}=4^7=16384$.

7. $z=-\frac{2}{3}\times\left(-\frac{1}{2}\right)^6=\frac{-2}{3\times 2^6}=\frac{-1}{3\times 2^5}=-\frac{1}{96}$.

CXLI.

1. $s=\frac{2(2^{15}-1)}{2-1}=2(32768-1)=65534$.

2. $s=\frac{1(3^6-1)}{3-1}=\frac{729-1}{2}=364$.

3. $s=\frac{a(x^{26}-1)}{x^2-1}$.

4. $s=\dfrac{a\left(\frac{1}{x^9}-1\right)}{\frac{1}{x}-1}=\frac{a}{x^8}\left(\frac{x^9-1}{x-1}\right)$.

5. $s=\dfrac{(a^2-x^2)\left\{\frac{1}{(a+x)^7}-1\right\}}{\frac{1}{a+x}-1}=\frac{(a^2-x^2)}{(a+x)^6}\left\{\frac{1-(a+x)^7}{1-a-x}\right\}$

$=\frac{(a-x)\{1-(a+x)^7\}}{(a+x)^5.(1-a-x)}$.

6. $s=\frac{2(3^n-1)}{3-1}=3^n-1$.

7. $s=\frac{7(2^n-1)}{2-1}=7(2^n-1)$.

$$s=\frac{5\{(-2)^8-1\}}{-2-1}=\frac{5(1-256)}{3}=-425.$$

$$s=\frac{-\frac{2}{3}\left\{\left(-\frac{1}{2}\right)^7-1\right\}}{-\frac{1}{2}-1}=\frac{-\frac{2}{3}\left\{-\frac{1}{128}-1\right\}}{-\frac{3}{2}}=\frac{4}{9}\times\left(-\frac{129}{128}\right)$$

$$=-\frac{43}{96}.$$

CXLII.

$s=\dfrac{1}{1-\frac{1}{2}}=\dfrac{1}{\frac{1}{2}}=2.$

2. $s=\dfrac{1}{1-\frac{1}{4}}=\dfrac{4}{3}.$

$s=\dfrac{3}{1-\frac{1}{9}}=\dfrac{3\times 9}{8}=\dfrac{27}{8}.$

4. $s=\dfrac{\frac{2}{3}}{1-\frac{1}{2}}=\dfrac{4}{3}.$

$s=\dfrac{\frac{3}{4}}{1-\frac{1}{3}}=\dfrac{3\times 3}{4\times 2}=1\dfrac{1}{8}.$

6. $s=\dfrac{\frac{1}{2}}{1+\frac{2}{3}}=\dfrac{1\times 3}{2\times 5}=\dfrac{3}{10}.$

$s=\dfrac{8}{1-\frac{1}{12}}=\dfrac{8\times 12}{11}=8\dfrac{8}{11}.$

8. $s=\dfrac{1\cdot 5}{1-\frac{1}{3}}=\dfrac{1\cdot 5\times 3}{2}=2\dfrac{1}{4}.$

$s=\dfrac{64}{1-\frac{1}{4}}=\dfrac{64\times 4}{3}=85\dfrac{1}{3}.$

10. $s=\dfrac{2x^3}{1+\frac{1}{8x^2}}=\dfrac{16x^5}{8x^2+1}.$

$s=\dfrac{a}{1-\frac{b}{a}}=\dfrac{a^2}{a-b}.$

12. $s=\dfrac{\frac{1}{10}}{1-\frac{1}{10}}=\dfrac{1}{9}.$

13. $s = \dfrac{x}{1+\dfrac{y}{x}} = \dfrac{x^2}{x+y}$.

14. $s = 86\left\{\dfrac{\dfrac{1}{100}}{1-\dfrac{1}{100}}\right\} = 86 \times \dfrac{1}{99} = \dfrac{86}{99}$.

15. $s = \dfrac{5}{10} + \dfrac{4}{100} + \dfrac{4}{1000} + \ldots = \dfrac{5}{10} + \dfrac{\dfrac{4}{100}}{1-\dfrac{1}{10}} = \dfrac{5}{10} + \dfrac{4}{90} = \dfrac{49}{90}$.

16. $s = \dfrac{8}{10} + \dfrac{36}{1000} + \dfrac{36}{100000} + \ldots = \dfrac{8}{10} + \dfrac{\dfrac{36}{1000}}{1-\dfrac{1}{100}} = \dfrac{8}{10} + \dfrac{36}{990} = \dfrac{46}{55}$.

CXLIII.

1. $243 = 3f^4$; $f^4 = 81$; $f = 3$, etc.

2. $1024 = 1 \times f^5$; $f^5 = 4^5$; $f = 4$, etc.

3. $16 = 1 \times f^4$; $f^4 = 16$; $f = 2$, etc.

4. $\dfrac{243}{64} = \dfrac{1}{2} \times f^5$; $f^5 = \dfrac{243}{32}$; $f = \dfrac{3}{2}$, etc.

CXLIV.

1. (1.) $s = \{16 + 7(12-1)\} \times \dfrac{12}{2} = 93 \times 6 = 558$.

(2.) $s = \{232 - 8(10-1)\} \times \dfrac{10}{2} = 160 \times 5 = 800$.

(3.) $s = \dfrac{3}{1-\dfrac{1}{6}} = \dfrac{18}{5}$.

(4.) $s=\dfrac{2}{1+\dfrac{1}{8}}=\dfrac{16}{9}$.

(5.) $s=\left\{1-\dfrac{7}{6}(13-1)\right\}\times\dfrac{13}{2}=(-13)\times\dfrac{13}{2}=-\dfrac{169}{2}$.

(6.) $s=\dfrac{\dfrac{1}{2}\cdot\left\{\dfrac{2^6}{3^6}-1\right\}}{-\dfrac{2}{3}-1}=\dfrac{3}{10}\cdot\left(\dfrac{3^6-2^6}{3^6}\right)=\dfrac{665}{2430}=\dfrac{133}{486}$.

(7.) $s=\left\{1-\dfrac{3}{2}\times 28\right\}\times\dfrac{29}{2}=(-41)\times\dfrac{29}{2}=-\dfrac{1189}{2}$.

(8.) $s=\left\{\dfrac{10}{7}+\dfrac{2}{7}\times 7\right\}\times\dfrac{8}{2}=\dfrac{24}{7}\times 4=13\dfrac{5}{7}$.

(9.) $s=\dfrac{\dfrac{1}{3}}{1-\dfrac{2}{3}}=\dfrac{1}{3}\times\dfrac{3}{1}=1$.

(10.) $s=\left\{\dfrac{6}{5}-2\times 9\right\}\times\dfrac{10}{2}=\left(-\dfrac{84}{5}\right)\times 5=-84$.

(11.) The common factor $=-\sqrt{6}\div\dfrac{\sqrt{3}}{\sqrt{5}}=-\dfrac{\sqrt{30}}{\sqrt{3}}=-\sqrt{10}$

$$s=\frac{\sqrt{\frac{3}{5}}\{(-\sqrt{10})^8-1\}}{-\sqrt{10}-1}=\frac{\sqrt{3}.\{10000-1\}}{\sqrt{5}\{-\sqrt{10}-1\}}=-\frac{9999\sqrt{3}}{\sqrt{5}(\sqrt{10}+1)}.$$

(12.) $s=\dfrac{-\dfrac{7}{5}\left\{-\dfrac{5^5}{2^5}-1\right\}}{-\dfrac{5}{2}-1}=-\dfrac{2}{5}\left(\dfrac{5^5+2^5}{2^5}\right)=-\dfrac{3157}{80}$.

2. Let the series be a, af, af^2, af^3, af^4.
Then $a\times af\times af^2\times af^3\times af^4=32$; $a^5f^{10}=2^5$; $af^2=2$.

3. $b=\dfrac{a+c}{2}$ and $b'=\sqrt{(ac)}$

$\therefore \dfrac{b}{b'}=\dfrac{a+c}{2\sqrt{(ac)}}$.

4. The arithmetic mean is $\frac{a+b}{2}$; the geometric is $\sqrt{(ab)}$.

Now since the square of every number is *positive*

$(\sqrt{a}-\sqrt{b})^2$ is greater than 0

$a-2\sqrt{(ab)}+b$ is greater than 0.

$\therefore \frac{a+b}{2}$ is greater than $\sqrt{(ab)}$.

5. $a+(a+d)+(a+2d)=12$; $3a+3d=12$; $a+d=4$.
Also, $a+5d=12$. Hence $4d=8$; $d=2$; $a=2$.
Then sum of 6 terms $=\{4+10\}\times 3=42$.

6. Let f be the common factor: $af=b$ and $af^2=c$
$\therefore ac=a^2f^2=b^2$.

7. $2n\times\frac{1}{2n}=x^2$; $1=x^2$; $x=\pm 1$.

8. $2n+\frac{1}{2n}=2y$; $y=n+\frac{1}{4n}$.

9. The sum of the geometric progression is $\frac{3^4-1}{3-1}$ or $\frac{80}{2}$ or 40.

The sum of the arithmetic progression is $\{8+4(n-1)\}\frac{n}{2}$.

Hence $2n+2n^2=40$; $n^2+n=20$; whence $n=4$.

10. The first term is 1, the constant difference 1.

Hence $153=\{2+(7+n-1)\}\frac{7+n}{2}$; $306=(8+n)(7+n)$;

$n^2+15n=250$; $n=10$.

11. Let n be the number of terms.

Then $\{2+2(n-1)\}\frac{n}{2}=2n\times\frac{n}{2}=n^2$.

12. Let the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$.
Then $5a+10d=95$; and $\therefore a+2d=19$.

3. $22=\left\{6\frac{2}{3}+\frac{13}{9}(n-1)\right\}\frac{n}{2}$; $44=n\left(\frac{47+13n}{9}\right)$;

$396=47n+13n^2$; whence $n=4$.

4. Let $100x+10y+z$ be the number.

Then $x+z=2y$ (1)

and $\frac{100x+10y+z}{y+z}=107$; or $100x-97y-106z=0$ (2)

and $100x+10y+z-396=100z+10y+x$. (3)

From (3) $99x-396=99z$; or $x-z=4$.

From this and (1) $x=y+2$ and $z=y-2$.

Hence from (2) $100y+200-97y-106y+212=0$.

Thus $y=4$, $x=6$, $z=2$.

5. In any geometrical series a, af, etc.

the $(p+q)$th term is af^{p+q-1}

the $(p-q)$th term is af^{p-q-1}

the pth term is af^{p-1}.

Hence $mn=af^{p+q-1}\times af^{p-q-1}=a^2f^{2p-2}=(af^{p-1})^2$

$\therefore af^{p-1}=\sqrt{(mn)}$.

6. Let x and y be the numbers. Then

$$\left.\begin{array}{l} x-y=48 \\ \frac{x+y}{2}=\sqrt{(xy)}+18 \end{array}\right\};\quad \left.\begin{array}{l} x-y=48 \\ x+y-2\sqrt{xy}=36 \end{array}\right\};\quad \left.\begin{array}{r} x-y=48 \\ \sqrt{x}-\sqrt{y}=6 \end{array}\right\};$$

dividing, $\sqrt{x}+\sqrt{y}=8$; hence $2\sqrt{x}=14$; $\sqrt{x}=7$; $x=49$; $y=1$

7. $11=1+4d$; $4d=10$; $d=\frac{5}{2}$, etc.

8. $2{\cdot}748=\{{\cdot}068+(n-1)\times{\cdot}0004\}\times\frac{n}{2}$; $5{\cdot}496={\cdot}0676n+{\cdot}0004n^2$;

$54960=4n^2+676n$; $n^2+169n=13740$; $n=60$.

9. $-1=1+10d$; $10d=-2$; $d=-\frac{1}{5}$, etc.

20. The $(n+1)$th term is 2^n.

The sum of n terms is $\dfrac{2^n-1}{2-1}=2^n-1$.

21. Sum of the *first* p terms $=\dfrac{a(r^p-1)}{r-1}$.

Sum of the *second* p terms $=\dfrac{ar^p(r^p-1)}{r-1}$.

Sum of the *third* p terms $=\dfrac{ar^{2p}(r^p-1)}{r-1}$, and so on, the sums forming a geometric series whose common factor is r^p.

22. $\left.\begin{array}{l}(a-2d)+(a-d)+a+(a+d)+(a+2d)=55\\(a-2d)^2+(a-d)^2+a^2+(a+d)^2+(a+2d)^2=765\end{array}\right\}$;

$\left.\begin{array}{l}5a=55\\5a^2+10d^2=765\end{array}\right\}$; $a=11$; $d=4$, etc.

23. Let the series be a, af, af^2, af^3, af^4.

$\left.\begin{array}{l}af^4-a:af^3-af=10:3\\af+af^3=2a^2f\end{array}\right\}$; $\left.\begin{array}{l}3f^4-3=10f^3-10f\\1+f^2=2a\end{array}\right\}$;

$\left.\begin{array}{l}3(f^4-1)=10f(f^2-1)\\1+f^2=2a\end{array}\right\}$; $\left.\begin{array}{ll}3(f^2+1)=10f & (1)\\1+f^2=2a & (2)\end{array}\right\}$;

from (1) we find $f=3$; and then from (2) $a=5$, etc.

24. This is explained in Art. 472.

25. Let $100x+10y+z$ be the number. Then

$$\left.\begin{array}{l}y^2=xz\\x+y+z=13\\100x+10y+z+792=100z+10y+x\end{array}\right\}$$

From the third equation we get $x-z=-8$.

Hence $x^2-2xz+z^2=64$.

But $\quad 4xz \quad =4y^2$

$\therefore\ (x+z)^2=64+4y^2$.

Also $x+z=13-y$

$\therefore\ (13-y)^2=64+4y^2$.

Hence we get $y=3$; and then $x=1$ and $z=9$.

6. Let x be the increase per cent. Then the population of any year is $\frac{100+x}{100}$ times the population of the preceding year.

Hence $10000 \times \left(\frac{100+x}{100}\right)^4 = 14641.$

Take the fourth root of each side; then

$$10 \times \frac{100+x}{100} = 11;\ 100+x=110;\ x=10.$$

CXLV.

1. First insert two arithmetic means between $\frac{1}{6}$ and $\frac{1}{24}$.

$$\frac{1}{24}=\frac{1}{6}+3d;\ 3d=-\frac{3}{24};\ d=-\frac{1}{24}.$$

Hence the arithmetic means are $\frac{1}{8}$ and $\frac{1}{12}$

$\therefore$ the harmonic means are 8 and 12.

2. $\frac{1}{3}=\frac{1}{2}+5d;\ 5d=-\frac{1}{6};\ d=-\frac{1}{30}.$

Hence the arithmetic means are $\frac{7}{15}, \frac{13}{30}, \frac{2}{5}, \frac{11}{30}$

$\therefore$ the harmonic means are $\frac{15}{7}, \frac{30}{13}, \frac{5}{2}, \frac{30}{11}.$

3. $\frac{2}{3}=3+4d;\ 4d=-\frac{7}{3};\ d=-\frac{7}{12}.$

Hence the arithmetic means are $\frac{29}{12}, \frac{11}{6}, \frac{5}{4}$

$\therefore$ the harmonic means are $\frac{12}{29}, \frac{6}{11}, \frac{4}{5}.$

4. $18=3+5d;\ 5d=15;\ d=3.$

Hence the arithmetic means are 6, 9, 12, 15

$\therefore$ the harmonic means are $\frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}.$

5. $2^{-1}=\frac{1}{2}$; and we have to insert five arithmetic means between -1 and 2.

$2=-1+6d$; $6d=3$; $d=\frac{1}{2}$.

Hence the arithmetic means are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$

$\therefore$ the harmonic means are -2, ∞, 2, 1, $\frac{2}{3}$.

6. $-2=2+6d$; $6d=-4$; $d=-\frac{2}{3}$.

Hence the arithmetic means are $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, $-\frac{4}{3}$

$\therefore$ the harmonic means are $\frac{3}{4}$, $\frac{3}{2}$, ∞, $-\frac{3}{2}$, $-\frac{3}{4}$.

7. $\frac{23}{6}=\frac{1}{3}+7d$; $7d=\frac{21}{6}$; $d=\frac{1}{2}$.

Hence the arithmetic means are $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, $\frac{7}{3}$, $\frac{17}{6}$, $\frac{10}{3}$

$\therefore$ the harmonic means are $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

8. $\frac{1}{3y}=\frac{1}{2x}+(n+1)d$; $d=\frac{2x-3y}{6xy(n+1)}$.

First arithmetic mean is $\frac{1}{2x}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+3y+2x-3y}{6xy(n+1)}$

$=\frac{3ny+2x}{6xy(n+1)}$.

Second arithmetic mean is $\frac{3ny+2x}{6xy(n+1)}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+4x-3y}{6xy(n+1)}$.

. . . .

nth arithmetic mean is $\frac{1}{3y}-\frac{2x-3y}{6xy(n+1)}=\frac{2nx+3y}{6xy(n+1)}$.

Hence the harmonic means are $\frac{6xy(n+1)}{3ny+2x}$, $\frac{6xy(n+1)}{3ny+4x-3y}$, . . .

$\frac{6xy(n+1)}{2nx+3y}$.

9. Let $\dfrac{1}{x}$ and $\dfrac{1}{y}$ be the second and third terms.

Then 2, x, y are in arithmetic progression.

$$\left.\begin{aligned}&\text{Hence } 2+y=2x\\ &\text{and } \frac{1}{2}+\frac{1}{x}+\frac{1}{y}=\frac{11}{12}\end{aligned}\right\};\quad \frac{1}{2}+\frac{1}{x}+\frac{1}{2x-2}=\frac{11}{12}$$

$$\frac{2x-2+x}{2x^2-2x}=\frac{5}{12};\ 36x-24=10x^2-10x;\ 10x^2-46x=-24;$$

hence we find $x=4$ or $\dfrac{3}{5}$; and $\therefore y=6$ or $-\dfrac{4}{5}$.

The arithmetic progressions will therefore be

$$-4,\ -2,\ 0,\ 2,\ 4,\ 6, \text{ and } \frac{31}{5},\ \frac{24}{5},\ \frac{17}{5},\ 2,\ \frac{3}{5},\ -\frac{4}{5}.$$

$\therefore$ the harmonic progressions are

$$-\frac{1}{4},\ -\frac{1}{2},\ \infty,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{6} \text{ and } \frac{5}{31},\ \frac{5}{24},\ \frac{5}{17},\ \frac{1}{2},\ \frac{5}{3},\ -\frac{5}{4}.$$

0. Let x and y be the numbers. Then

$$\frac{x+y}{2}=\sqrt{(xy)}+13\ \ (1);\ \text{and } \sqrt{(xy)}=\frac{2xy}{x+y}+12$$

$$\frac{2xy}{x+y}=\sqrt{(xy)}-12\ \ (2).$$

Multiply (1) by (2); $xy=xy+\sqrt{(xy)}-156$.

Hence $\sqrt{(xy)}=156$; from this and (1) we find $x=104$, $y=234$.

1. $2b=a+c$ and $\dfrac{1}{b}+\dfrac{1}{d}=\dfrac{2}{c}$

$$\therefore 2b\left(\frac{1}{b}+\frac{1}{d}\right)=\frac{2}{c}(a+c);\ 2+\frac{2b}{d}=\frac{2a}{c}+2;$$

$\therefore bc=ad$, and $\therefore a:b=c:d$.

2. $$x=\frac{2mn}{m+n};\ \frac{1}{x-m}+\frac{1}{x-n}=\frac{1}{\dfrac{2mn}{m+n}-m}+\frac{1}{\dfrac{2mn}{m+n}-n}$$

$$=\frac{m+n}{mn-m^2}+\frac{m+n}{mn-n^2}=\frac{m+n}{n-m}\left\{\frac{1}{m}-\frac{1}{n}\right\}=\frac{m+n}{mn}=\frac{1}{m}+\frac{1}{n}.$$

13. $\left.\begin{aligned} x+y+z&=11 \\ x^2+y^2+z^2&=49 \\ y&=\frac{2xz}{x+z} \end{aligned}\right\};\quad \left.\begin{aligned} (x+z)^2&=121-22y+y^2 \\ x^2+z^2&=49-y^2 \\ 2xz&=72-22y+2y^2 \end{aligned}\right\}$

Now $(x+z)y=2xz$.

$\therefore\ (11-y)y=72-22y+2y^2$; whence $y=3$.

Then $x+z=8$, and $xz=12$; whence $x=2$, $z=6$.

14. $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are the pth, qth, and rth terms of an arithmetic progression, suppose $a, a+d \ldots$

Then $\left.\begin{aligned} \frac{1}{x}&=a+(p-1)d \\ \frac{1}{y}&=a+(q-1)d \\ \frac{1}{z}&=a+(r-1)d \end{aligned}\right\};\quad \begin{aligned} \frac{1}{x}-\frac{1}{z}&=(p-r)d \\ \frac{1}{z}-\frac{1}{y}&=(r-q)d \\ \frac{1}{y}-\frac{1}{x}&=(q-p)d \end{aligned}$

Hence $\left.\begin{aligned} z-x&=(p-r)xzd \\ y-z&=(r-q)yzd \\ x-y&=(q-p)xyd \end{aligned}\right\};$

adding $0=(p-r)xzd+(r-q)yzd+(q-p)xyd$

$\therefore\quad 0=(p-r)xz+(r-q)yz+(q-p)xy.$

15. $\frac{2ab}{a+b}, \frac{2bc}{b+c}, \frac{2ca}{c+a}$ are in A. P.

$\therefore\ \frac{bc}{b+c}-\frac{ab}{a+b}=\frac{ca}{c+a}-\frac{bc}{b+c}$

$\therefore\ \frac{b^2c-ab^2}{(b+c)(a+b)}=\frac{c^2a-bc^2}{(b+c)(c+a)}$, or, $\frac{b^2(c-a)}{a+b}=\frac{c^2(a-b)}{c+a}$

$\therefore\ b^2(c^2-a^2)=c^2(a^2-b^2)$, or, $2b^2c^2=a^2b^2+a^2c^2$

$\therefore\ \frac{2}{a^2}=\frac{1}{c^2}+\frac{1}{b^2}$; $\because\ b^2, a^2, c^2$ are in H. P.

Again, $\frac{a+b}{2ab}, \frac{b+c}{2bc}, \frac{c+a}{2ca}$ are in A. P.

$$\therefore \frac{b+c}{bc}-\frac{a+b}{ab}=\frac{c+a}{ac}-\frac{b+c}{bc}$$

$$\therefore \frac{1}{c}+\frac{1}{b}-\frac{1}{b}-\frac{1}{a}=\frac{1}{a}+\frac{1}{c}-\frac{1}{c}-\frac{1}{b}$$

$$\therefore \frac{1}{c}+\frac{1}{b}=\frac{2}{a};\ \therefore b, a, c \text{ are in H. P.}$$

6. (1.) When c is the arithmetic mean between a and b.

$$\frac{\frac{a+b}{2}+2a}{\frac{a+b}{2}-b}+\frac{\frac{a+b}{2}+2b}{\frac{a+b}{2}-a}=\frac{5a+b}{a-b}+\frac{a+5b}{b-a}$$

$$=\frac{5a+b-a-5b}{a-b}=\frac{4a-4b}{a-b}=4.$$

(2.) When c is the geometric mean between a and b.

$$\frac{\sqrt{(ab)}+2a}{\sqrt{(ab)}-b}+\frac{\sqrt{(ab)}+2b}{\sqrt{(ab)}-a}$$

$$=\frac{\sqrt{a}(\sqrt{b}+2\sqrt{a})}{\sqrt{b}(\sqrt{a}-\sqrt{b})}-\frac{\sqrt{b}(\sqrt{a}+2\sqrt{b})}{\sqrt{a}(\sqrt{a}-\sqrt{b})}=\frac{a\sqrt{b}+2a\sqrt{a}-b\sqrt{a}-2b\sqrt{b}}{\sqrt{(ab)}.(\sqrt{a}-\sqrt{b})}$$

$$=\frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}};\ \text{and since } a+b \text{ is greater than } 2\sqrt{(ab)},$$

$$\therefore \frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}} \text{ is greater than } \frac{4\sqrt{(ab)}+3\sqrt{(ab)}}{\sqrt{(ab)}}, \text{ or, than } 7.$$

(3.) When c is the harmonic mean between a and b.

$$\frac{\frac{2ab}{a+b}+2a}{\frac{2ab}{a+b}-b}+\frac{\frac{2ab}{a+b}+2b}{\frac{2ab}{a+b}-a}=\frac{4ab+2a^2}{ab-b^2}+\frac{4ab+2b}{ab-a^2}$$

$$=\frac{4a^2b+2a^3-4ab^2-2b^3}{ab(a-b)}=\frac{2a^2+6ab+2b^2}{ab}.$$

Now a^2+b^2 is greater than $2ab$

$$\therefore \frac{2a^2+2b^2+6ab}{ab} \text{ is greater than } \frac{4ab+6ab}{ab}, \text{ or, than } 10.$$

CXLVI.

1. $n.(n-1)=12\times 11=132.$

2. $n.(n-1).(n-2)=16\times 15\times 14=3360.$

3. $n.(n-1).(n-2).(n-3)=20\times 19\times 18\times 17=116280.$

4. $n.(n-1).(n-2)(n-3)(n-4)=8\times 7\times 6\times 5\times 4=6720.$

5. Number of letters is 11, and three are twice repeated,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|2}\,\underline{|2}\,\underline{|2}}=4989600.$

6. $n.(n-1)\ .\ .\ .\ (n-7)=8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=40320.$

7. $n.(n-1)\ .\ .\ .\ (n-9)=\underline{|10}=3628800.$

8. (1.) The number of signals in which we have 3 flags of different colours is $5.4.3=60$.

(2.) When 2 flags in a signal are of the same colour, suppose red, we can put 4 different colours with these, and each of the resulting signals may be arranged in 3 different ways, that is, with both the red first, or both last, or one first and one last. Thus we shall have 12 different signals with two reds. Similarly we shall have 12 different signals with two of each of the other 4 colours.

$\therefore$ we shall have 5×12 or 60 in all.

(3.) When we have 3 of the same colour, we shall have 5 different signals in all.

$\therefore$ total number $=60+60+5=125.$

9. Number of permutations $=\dfrac{|7}{\underline{|2}}=\dfrac{5040}{2}=2520.$

10. $n:n(n-1)(n-2)=1:20$;

$20n=n(n-1)(n-2)$; $n^2-3n+2=20$; $\therefore n=6.$

11. $m.(m-1)(m-2):(m+2)(m+1)m=1:5$;

$5m.(m-1)(m-2)=(m+2)(m+1)m$;

$5m^2-15m+10=m^2+3m+2$; $m=4$.

12. Number of letters is 7, and therefore the number of permutations in which *cd* stand first $=\underline{|5}=120$.

13. Number of letters is 9, one of which is repeated twice, one three times, and one four times.

$$\therefore \text{ Number of permutations } = \frac{\underline{|9}}{\underline{|2}.\underline{|3}.\underline{|4}} = 9\times4\times7\times5=1260.$$

14. In *Conceit* we have 7 letters, one of which occurs twice,

$$\therefore \text{ number of permutations } = \frac{\underline{|7}}{\underline{|2}} = 2520.$$

In *Talavera*, 8 letters, one of which is repeated three times,

$$\therefore \text{ number of permutations } = \frac{\underline{|8}}{\underline{|3}} = 6720.$$

In *Calcutta*, 8 letters, 3 of them repeated twice,

$$\therefore \text{ number of permutations } = \frac{\underline{|8}}{2\times2\times2} = 5040.$$

In *Proposition*, 11 letters, 2 repeated twice, 1 thrice,

$$\therefore \text{ number of permutations } = \frac{\underline{|11}}{\underline{|2}.\ \underline{|2}.\ \underline{|3}} = 1663200.$$

In *Mississippi*, 11 letters, 2 repeated 4 times, 1 twice,

$$\therefore \text{ number of permutations } = \frac{\underline{|11}}{\underline{|4}.\ \underline{|4}.\ \underline{|2}} = 34650.$$

CXLVII.

1. $${}_{100}C_4 = \frac{100\times99\times98\times97}{1\times2\times3\times4} = 3921225.$$

2. $${}_6C_5 = \frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5} = 6.$$

3. Leave out a, and find how many combinations can be formed out of the remaining 9 letters, 4 at a time

$_9C_4=\frac{9\times8\times7\times6\times5}{1\times2\times3\times4\times5}=126$, with each of which a will combine to form a word of 5 letters.

4. Out of the 19 consonants we get $\frac{19\times18\times17}{1\times2\times3}$ combinations of 3,

then $\frac{19\times18\times17}{1\times2\times3}\times5$ = number of ways in which we can get 3 consonants and 1 vowel, and each combination of 4 letters admits of $\underline{|4}$ permutations.

$\therefore$ number of words $=\frac{19.18.17}{1.2.3}\times5\times\underline{|4}=116280.$

5. $\frac{n.(n-1)(n-2)(n-3)}{1.2.3.4}:\frac{n.(n-1)}{1.2}=15:2$; $(n-2)(n-3)=90$;

$n=12.$

6. $\frac{n.(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\frac{18}{5}\times\frac{n.(n-1)(n-2)}{1.2.3}$;

$(n-3)(n-4)=18\times4$; $n=12.$

7. Number of words $=\frac{17.16.15}{1.2.3}\times\frac{5.4}{1.2}\times\underline{|5}=816000.$

8. Number $=\frac{12.11.10.9.8.7}{1.2.3.4.5.6}\times\frac{5.4.3}{1.2.3}\times\underline{|9}=3353011200.$

9. $n(n-1)(n-2)=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\times6$; $6(n-3)=24$;

$n-3=4$; $n=7.$

10. Taking the coins singly, by twos, by threes, and so on,

number of combinations $=6+\frac{6.5}{1.2}+\frac{6.5.4}{1.2.3}+\frac{6.5.4.3}{1.2.3.4}+\frac{6.5.4.3.2}{1.2.3.4.5}+1$

$=6+15+20+15+6+1=63.$

11. $\frac{n.(n-1)(n-2)}{1.2.3}=425\times n$; $(n-1)(n-2)=2550$; hence $n=52.$

12. Number $=\dfrac{12.11.10}{1.2.3}\times\dfrac{16.15.14}{1.2.3}=123200.$

13. (1.) The number is $\dfrac{36.35.34.33.32}{1.2.3.4.5}=376992.$

(2.) He will go out with as many different parties as can be formed by taking 4 out of 35.

$\therefore$ he goes with $\dfrac{35.34.33.32}{1.2.3.4}=52360.$

CXLVIII.

1. $a^4+4a^3x+\dfrac{4.3}{1.2}a^2x^2+\dfrac{4.3.2}{1.2.3}ax^3+x^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

2. $b^6+6b^5c+\dfrac{6.5}{1.2}b^4c^2+\dfrac{6.5.4}{1.2.3}b^3c^3+\dfrac{6.5.4.3}{1.2.3.4}b^2c^4+\dfrac{6.5.4.3.2}{1.2.3.4.5}bc^5+c^6$
$=b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6.$

3. $a^7+7a^6b+\dfrac{7.6}{1.2}a^5b^2+\dfrac{7.6.5}{1.2.3}a^4b^3+\dfrac{7.6.5.4}{1.2.3.4}a^3b^4+\dfrac{7.6.5.4.3}{1.2.3.4.5}a^2b^5$
$+\dfrac{7.6.5.4.3.2}{1.2.3.4.5.6}ab^6+b^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$
$+21a^2b^5+7ab^6+b^7.$

4. $x^8+8x^7y+\dfrac{8.7}{1.2}x^6y^2+\dfrac{8.7.6.}{1.2.3.}x^5y^3+\dfrac{8.7.6.5}{1.2.3.4}x^4y^4+\dfrac{8.7.6.5.4}{1.2.3.4.5}x^3y^5$
$+\dfrac{8.7.6.5.4.3}{1.2.3.4.5.6}x^2y^6+\dfrac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}xy^7+y^8$
$=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8.$

5. $5^4+4.5^3.4a+\dfrac{4.3}{1.2}.5^2.4^2a^2+\dfrac{4.3.2}{1.2.3}.5.4^3a^3+(4a)^4$
$=625+2000a+2400a^2+1280a^3+256a^4.$

6. $(a^2)^5+5(a^2)^4.bc+\dfrac{5.4}{1.2}(a^2)^3.(bc)^2+\dfrac{5.4.3}{1.2.3}(a^2)^2.(bc)^3+\dfrac{5.4.3.2}{1.2.3.4}a^2(bc)^4$
$+(bc)^5=a^{10}+5a^8bc+10a^6b^2c^2+10a^4b^3c^3+5a^2b^4c^4+b^5c^5.$

CLI.

1. The 8th term of $(1+x)^{11}$ is $\dfrac{11.10.9.8.7.6.5}{1.2.3.4.5.6.7}\cdot x^7 = 330x^7$.

2. The 5th term of $(a^2-b^2)^{12}$ is $\dfrac{12.11.10.9}{1.2.3.4}\cdot(a^2)^{12-5+1}.(b^2)^4 = 495a^{16}.b^8$.

3. The 4th term of $(a-b)^{100}$ is $-\dfrac{100.99.98}{1.2.3}\cdot a^{100-4+1}.b^3 = -161700a^{97}b^3$.

4. The 9th term of $(2ab-cd)^{14}$ is $\dfrac{14.13.12.11.10.9.8.7}{1.2.3.4.5.6.7.8}\cdot(2ab)^{14-9+1}.(cd)^8$

   $=192192a^6b^6c^8d^8$.

5. The 9th term of $(a-b)^{16}$ is $\dfrac{16.15.14.13.12.11.10.9}{1.2.3.4.5.6.7.8}\cdot a^{16-9+1}.b^8$

   $=12870a^8b^8$.

6. The 5th term of $(a^{\frac{1}{8}}+b^{\frac{1}{8}})^8$ is $\dfrac{8.7.6.5}{1.2.3.4}\cdot(a^{\frac{1}{8}})^{8-5+1}.(b^{\frac{1}{8}})^4 = 70a^{\frac{1}{2}}b^{\frac{1}{2}}$.

7. The 10th term of $(a-b)^{19}$ is $-\dfrac{19.18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8.9}\cdot a^{10}.b^9$

   $=-92378a^{10}b^9$.

   The 11th term of $(a-b)^{19}$ is therefore $92378a^9b^{10}$.

8. The 7th term of $(a+x)^{13}$ is $\dfrac{13.12.11.10.9.8}{1.2.3.4.5.6}\cdot a^7.x^6 = 1716a^7x^6$.

   The 8th term of $(a+x)^{13}$ is therefore $1716a^6x^7$.

9. The coefficient of the rth term of $(a+x)^n$ is $\dfrac{n.(n-1)\ldots(n-r+2)}{1.2\ .\ .\ .\ (r-1)}$,

   and the middle term of $(a+x)^{4n}$ is the $(2n+1)$th.

$$\therefore \text{ Coefficient } = \frac{4n.(4n-1)\ .\ .\ .\ (4n-2n-1+2)}{1.2\ .\ .\ .\ (2n+1-1)}$$

$$= \frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}\cdot\frac{2n.(2n-1)\ .\ .\ .\ 1}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-2)\ .\ .\ .\ 6.4.2}{2n.(2n-1)\ .\ .\ .\ 3.2.1}\cdot\frac{(4n-1)(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2.2\ .\ .\ .\ \text{to } 2n \text{ factors}.\frac{(4n-1).(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2^{2n}.\ \frac{1.3.5\ .\ .\ .\ (4n-1)}{1.2.3\ .\ .\ .\ 2n}.$$

ɔ. The middle term is the $(2n+2)$th.

$$\therefore \text{ Coefficient } =\frac{(4n+2)(4n+1)\ .\ .\ .\ (4n+2-2n-2+2)}{1.2\ .\ .\ .\ (2n+2-1)}$$

$$=\frac{(4n+2).(4n+1)\ .\ .\ .\ (2n+2)}{1.2\ .\ .\ .\ (2n+1)}$$

$$=\frac{(4n+2)(4n)(4n-2)...(2n+4)(2n+2)}{(2n+1)(2n)(2n-1)...(n+1)}\cdot\frac{(4n+1)(4n-1)...(2n+3)}{n.(n-1)...2.1}$$

$$=2.2\ .\ .\ .\ \text{to } (n+1) \text{ factors}.\frac{(4n+1)(4n-1)\ .\ .\ .\ (2n+3)}{n.(n-1)\ .\ .\ .\ 2.1}$$

$$=2^{n+1}.\frac{(2n+3)(2n+5)\ .\ .\ .\ (4n-1)(4n+1)}{1.2\ .\ .\ .\ n}.$$

CLII.

$$1.\ \ 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}x^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^4$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4.$$

2. $$1+\frac{2}{3}a+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}a^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}a^3$$

$$=1+\frac{2a}{3}-\frac{a^2}{9}+\frac{4a^3}{81}.$$

3. $$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\cdot\frac{x^3}{a^3}\right.$$

$$\left.+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1.2.3.4}\cdot\frac{x^4}{a^4}\right\}$$

$$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}.$$

4. $$1+x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}(2x)^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}(2x)^4$$

$$=1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}.$$

5. $$a^{\frac{3}{4}}\left\{1+\frac{x}{a}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{4x}{3a}\right)^2+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{4x}{3a}\right)^3\right\}$$

$$=a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}x^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}.x^3.$$

6. $$a^{\frac{2}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)}{1.2}\cdot\frac{x^{\frac{2}{2}}}{a^{\frac{2}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)\cdot\left(\frac{4}{5}-2\right)}{1.2.3}\cdot\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}\right\}$$

$$=a^{\frac{2}{5}}+\frac{4}{5}\cdot a^{-\frac{1}{10}}.x^{\frac{1}{2}}-\frac{2}{25}a^{-\frac{6}{10}}.x^{\frac{2}{2}}+\frac{4}{125}\cdot a^{-\frac{11}{10}}.x^{\frac{3}{2}}.$$

7. $1-\dfrac{x^2}{2}+\dfrac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1.2}\cdot x^4-\dfrac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)}{1.2.3}x^6$

$+\dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^8$

$=1-\dfrac{x^2}{2}-\dfrac{x^4}{8}-\dfrac{x^6}{16}-\dfrac{5x^8}{128}.$

8. $1-\dfrac{7a^2}{3}+\dfrac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)}{1.2}a^4-\dfrac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{1.2.3}a^6$

$=1-\dfrac{7a^2}{3}+\dfrac{14a^4}{9}-\dfrac{14a^6}{81}.$

9. $1-\dfrac{9x}{4}+\dfrac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}(3x)^2-\dfrac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot(3x)^3$

$=1-\dfrac{9x}{4}-\dfrac{27x^2}{32}-\dfrac{135x^3}{128}.$

10. $x^3\left\{1-\dfrac{3}{2}\cdot\dfrac{2y}{3x^2}+\dfrac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)}{1.2}\cdot\dfrac{4y^2}{9x^4}-\dfrac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1.2.3}\cdot\dfrac{8y^3}{27x^6}\right\}$

$=x^3-xy+\dfrac{y^2}{6x}+\dfrac{y^3}{54x^3}.$

11. $1-\dfrac{5}{6}x+\dfrac{\frac{5}{6}\cdot\left(\frac{5}{6}-1\right)}{1.2}x^2-\dfrac{\frac{5}{6}\left(\frac{5}{6}-1\right)\left(\frac{5}{6}-2\right)}{1.2.3}x^3$

$=1-\dfrac{5}{6}x-\dfrac{5}{72}x^2-\dfrac{35}{1296}x^3.$

12. $\left(\dfrac{2x}{3}\right)^{\frac{2}{3}}\left\{1-\dfrac{2}{3}\cdot\dfrac{9y}{4x}+\dfrac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{1.2}\cdot\dfrac{81y^2}{16x^2}\right\}$

$=\left(\dfrac{2}{3}\right)^{\frac{2}{3}}.x^{\frac{2}{3}}-\left(\dfrac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{1}{3}}y-\dfrac{3}{8}\left(\dfrac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{4}{3}}y^2.$

2. $1+\frac{2}{3}a+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}a^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}a^3$

$=1+\frac{2a}{3}-\frac{a^2}{9}+\frac{4a^3}{81}.$

3. $a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\cdot\frac{x^3}{a^3}\right.$

$\left.+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1.2.3.4}\cdot\frac{x^4}{a^4}\right\}$

$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}.$

4. $1+x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}(2x)^3$

$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}(2x)^4$

$=1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}.$

5. $a^{\frac{3}{4}}\left\{1+\frac{x}{a}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{4x}{3a}\right)^2+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{4x}{3a}\right)^3\right\}$

$=a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}a^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}.x^3.$

6. $a^{\frac{2}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)}{1.2}\cdot\frac{x^{\frac{2}{2}}}{a^{\frac{2}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)\cdot\left(\frac{4}{5}-2\right)}{1.2.3}\cdot\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}\right\}$

$=a^{\frac{2}{5}}+\frac{4}{5}\cdot a^{-\frac{1}{10}}.x^{\frac{1}{2}}-\frac{2}{25}a^{-\frac{6}{10}}.x^{\frac{2}{2}}+\frac{4}{125}\cdot a^{-\frac{11}{10}}.x^{\frac{3}{2}}.$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}+\frac{3x^4}{8}-\frac{5x^6}{16}+\frac{35x^8}{128}.$$

2. $$1+\frac{3}{2}x^2+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2}x^4-\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)\cdot\left(-\frac{9}{2}\right)}{1.2.3.4}x^8$$

$$=1+\frac{3x^2}{2}+\frac{15x^4}{8}+\frac{35x^6}{16}+\frac{315x^8}{128}.$$

3. $$x^{-2}\left\{1-\frac{2}{5}\cdot\frac{z^5}{x^5}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)}{1.2}\cdot\frac{z^{10}}{x^{10}}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)\cdot\left(-\frac{12}{5}\right)}{1.2.3}\frac{z^{15}}{x^{15}}\right\}$$

$$=x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{7}{25}x^{-12}z^{10}-\frac{28}{125}x^{-17}.z^{15}.$$

4. $$1-x+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}4x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}8x^3$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}16x^4$$

$$=1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{8}.$$

5. $$a^{-1}\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}\cdot\frac{x^4}{a^4}\right.$$

$$\left.+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}\cdot\frac{x^6}{a^6}\right\}$$

$$=\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}.$$

CLIII.

1. $1-2a+\frac{(-2)(-2-1)}{1.2}a^2+\frac{(-2)(-2-1)(-2-2)}{1.2.3}a^3+$

$\frac{(-2)(-2-1)(-2-2)(-2-3)}{1.2.3.4}a^4=1-2a+3a^2-4a^3+5a^4.$

2. $1+3x+\frac{(-1)\cdot(-2)}{1.2}(3x)^2-\frac{(-1).(-2).(-3)}{1.2.3}(3x)^3+$

$\frac{(-1).(-2).(-3).(-4)}{1.2.3.4}(3x)^4=1+3x+9x^2+27x^3+81x^4.$

3. $1+x+\frac{(-4).(-5)}{1.2}\cdot\frac{x^2}{16}-\frac{(-4).(-5).(-6)}{1.2.3}\cdot\frac{x^3}{64}$

$=1+x+\frac{5x^2}{8}+\frac{5x^3}{16}.$

4. $1+x+\frac{(-2).(-3)}{1.2}\cdot\frac{x^2}{4}-\frac{(-2).(-3).(-4)}{1.2.3}\cdot\frac{x^3}{8}+$

$\frac{(-2).(-3).(-4).(-5)}{1.2.3.4}\cdot\frac{x^4}{16}=1+x+\frac{3x^2}{4}+\frac{x^3}{2}+\frac{5x^4}{16}.$

5. $a^{-10}-(-5)a^{-12}.2x+\frac{(-5)(-6)}{1.2}a^{-14}.4x^2-\frac{(-5).(-6)(-7)}{1.2.3}.a^{-16}.8x^3$

$+\frac{(-5).(-6).(-7).(-8)}{1.2.3.4}a^{-18}.16x^4$

$=a^{-10}+10a^{-12}x+60a^{-14}x^2+280a^{-16}.x^3+1120a^{-18}x^4.$

6. $a^{-2}-(-6)a^{-\frac{7}{3}}\cdot x^{\frac{1}{3}}+\frac{(-6)\cdot(-7)}{1.2}\cdot a^{-\frac{8}{3}}\cdot x^{\frac{2}{3}}-\frac{(-6)\cdot(-7)\cdot(-8)}{1.2.3}\cdot a^{-3}x$

$=\frac{1}{a^2}+\frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}}+\frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}}+\frac{56x}{a^3}.$

CLIV.

1. $1-\frac{1}{2}x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}x^4+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}x^6$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}+\frac{3x^4}{8}-\frac{5x^6}{16}+\frac{35x^8}{128}.$$

2. $$1+\frac{3}{2}x^2+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2}x^4-\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)\cdot\left(-\frac{9}{2}\right)}{1.2.3.4}x^8$$

$$=1+\frac{3x^2}{2}+\frac{15x^4}{8}+\frac{35x^6}{16}+\frac{315x^8}{128}.$$

3. $$x^{-2}\left\{1-\frac{2}{5}\cdot\frac{z^5}{x^5}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)}{1.2}\cdot\frac{z^{10}}{x^{10}}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)\cdot\left(-\frac{12}{5}\right)}{1.2.3}\frac{z^{15}}{x^{15}}\right\}$$

$$=x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{7}{25}x^{-12}z^{10}-\frac{28}{125}x^{-17}.z^{15}.$$

4. $$1-x+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}4x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}8x^3$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}16x^4$$

$$=1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{8}.$$

5. $$a^{-1}\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}\cdot\frac{x^4}{a^4}\right.$$

$$\left.+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}\cdot\frac{x^6}{a^6}\right\}$$

$$=\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}.$$

6. $a^{-1}\left\{1-\frac{1}{3}\cdot\frac{x^3}{a^3}+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)}{1.2}\cdot\frac{x^6}{a^6}\right.$

$\left.+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)}{1.2.3}\cdot\frac{x^9}{a^9}\right\}=\frac{1}{a}-\frac{x^3}{3a^4}+\frac{2x^6}{9a^7}-\frac{14}{81a}$

CLV.

The rth term of $(a+x)^n$ is $\frac{n.(n-1)\,.\,.\,.\,(n-r+2)}{1.2\,.\,.\,.\,(r-1)}x^{r-1}.\,a^{n-r+1}$

1. The rth term is $\frac{7\cdot 6\,.\,.\,.\,(9-r)}{1\cdot 2\,.\,.\,.\,(r-1)}x^{r-1}$.

2. The rth term is $(-1)^{r-1}\cdot\frac{12.11\,.\,.\,.\,(14-r)}{1.2\,.\,.\,.\,(r-1)}x^{r-1}$.

3. The rth term is $(-1)^{r-1}\cdot\frac{8.7\,.\,.\,.\,(10-r)}{1.2\,.\,.\,.\,(r-1)}a^{9-r}\cdot x^{r-1}$.

4. The rth term is $\frac{9.8\,.\,.\,.\,(11-r)}{1.2\,.\,.\,.\,(r-1)}\cdot(5x)^{10-r}\cdot(2y)^{r-1}$.

5. The rth term is $\frac{(-2)\cdot(-3)\,.\,.\,.\,(-2-r+2)}{1.2\,.\,.\,.\,(r-1)}x^{r-1}$.

$$=(-1)^{r-1}\cdot\frac{2.3\,.\,.\,.\,r}{1.2\,.\,.\,.\,(r-1)}\cdot x^{r-1}=(-1)^{r-1}\cdot r\cdot x^{r-1}.$$

6. The rth term is $\frac{4.5\,.\,.\,.\,(r+2)}{1.2\,.\,.\,.\,(r-1)}\cdot(3x)^{r-1}$

$$=\frac{r\cdot(r+1)\cdot(r+2)}{1.2.3}\cdot(3x)^{r-1}.$$

7. The rth term is $\frac{\frac{1}{2}\cdot\frac{3}{2}\,.\,.\,.\,\left(r-\frac{3}{2}\right)}{1.2\,.\,.\,.\,(r-1)}\cdot x^{r-1}$

$$=\frac{\left\{1.3.5\ .\ .\ .\ (2r-3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}$$

$$=\frac{1.3.5\ .\ .\ .\ (2r-3)}{1.2\ .\ .\ .\ (r-1)}\cdot\left(\frac{x}{2}\right)^{r-1}.$$

8. The rth term is $(-1)^{r-1}\cdot\dfrac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\ .\ .\ .\ \left(r-\frac{7}{3}\right)}{1.2\ .\ .\ .\ (r-1)}\cdot a^{\frac{1}{3}-r+1}x^{r-1}$

$$=(-1)^{r-1}\cdot\frac{\left\{1.2.5\ .\ .\ .\ (3r-7)\right\}\times\left(\frac{1}{3}\right)^{r-1}}{1.2\ .\ .\ .\ (r-1)}.a^{\frac{1}{3}}\cdot\left(\frac{x}{a}\right)^{r-1}$$

$$=(-1)^{r-1}\cdot\frac{1.2.5\ .\ .\ .\ (3r-7)}{1.2\ .\ .\ .\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(\frac{x}{3a}\right)^{r-1}$$

$$=\frac{1.2.5\ .\ .\ .\ (3r-7)}{1.2\ .\ .\ .\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(-\frac{x}{3a}\right)^{r-1}.$$

9. The rth term is $\dfrac{\frac{7}{2}\cdot\left(\frac{7}{2}+1\right)\ .\ .\ .\ \left(\frac{7}{2}+r-2\right)}{1.2\ .\ .\ .\ (r-1)}\cdot(2x)^{r-1}$

$$=\frac{\left\{7.9\ .\ .\ .\ (2r+3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ .\ .\ .\ (r-1)}\cdot(2x)^{r-1}$$

$$=\frac{7.9\ .\ .\ .\ (2r+3)}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}.$$

10. The rth term is $\dfrac{\frac{3}{4}\cdot\left(\frac{3}{4}+1\right)\ .\ .\ .\ \left(\frac{3}{4}+r-2\right)}{1.2\ .\ .\ .\ (r-1)}.(a^2)^{-\frac{3}{4}-r+1}.(x^2)^{r-1}$

$$=\frac{\{3.7\ .\ .\ .\ (4r-5)\}\times\left(\frac{1}{4}\right)^{r-1}}{1.2\ .\ .\ .\ (r-1)}.a^{-\frac{3}{2}}.\left(\frac{x^2}{a^2}\right)^{r-1}$$

$$=\frac{3.7\ .\ .\ .\ (4r-5)}{1.2\ .\ .\ .\ (r-1)}\cdot\frac{a^{-\frac{3}{2}}}{4^{r-1}}\cdot\left(\frac{x}{a}\right)^{2(r-1)}$$

11. The $(r+1)$th term is $\frac{3.4\ \ldots\ (3+r-1)}{1.2\ \ldots\ r}x^r$

$$=\frac{3.4\ \ldots\ (2+r)}{1.2\ \ldots\ r}.x^r=\frac{(r+1)(r+2)}{2}.x^r.$$

12. The $(r+1)$th term is $\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\ \ldots\ \left(\frac{1}{2}+r-1\right)}{1.2\ \ldots\ r}(4x)^r$

$$=\frac{\{1.3\ \ldots\ (2r-1)\}\times\left(\frac{1}{2}\right)^r}{1.2\ \ldots\ r}.(4x)^r$$

$$=\frac{1.3\ \ldots\ (2r-1)}{1.2\ \ldots\ r}.(2x)^r.$$

13. The $(r+1)$th term is $\frac{2r.(2r-1)\ \ldots\ (r+1)}{1.2\ \ldots\ r}x^r$

$$=\frac{2r.(2r-1)\ \ldots\ (r+1).r.(r-1)\ \ldots\ 2.1}{(\underline{|r})^2}.x^r$$

$$=\frac{(2r-1).(2r-3)\ldots 3.1}{\underline{|r}}.\frac{2r.(2r-2)\ldots 4.2}{\underline{|r}}.x^r$$

$$=\frac{(2r-1)(2r-3)\ \ldots\ 3.1}{\underline{|r}}.\frac{2^r.(\underline{|r})}{\underline{|r}}.x^r$$

$$=\frac{1.3.5\ \ldots\ (2r-1)}{1.2.3\ \ldots\ r}.(2x)^r.$$

14. Coefficient of x^{r+1} in $(1+x)^{n+1}$ is $\frac{(n+1.)n.(n-1)\ \ldots\ (n-r+1)}{1.2.3\ \ldots\ (r+1)}$

Coefficient of x^r in $(1+x)^n$ is $\frac{n.(n-1)\ \ldots\ (n-r+1)}{1.2.3\ \ldots\ r}$.

Coefficient of x^{r+1} in $(1+x)^n$ is $\frac{n.(n-1)\ \ldots\ (n-r)}{1.2\ \ldots\ (r+1)}$.

Now $\frac{(n+1)n(n-1)\ldots(n-r+1)}{1.2.3\ldots(r+1)}=\frac{n.(n-1)\ldots(n-r+1)}{1.2\ \ldots\ r}.\frac{n+1}{r+1}$

$$=\frac{n.(n-1)\ \ldots\ (n-r+1)}{1.2\ \ldots\ r}.\left(1+\frac{n-r}{r+1}\right)$$

$$=\frac{n.(n-1)\ \ldots\ (n-r+1)}{1.2\ \ldots\ r}+\frac{n.(n-1)\ \ldots\ (n-r)}{1.2\ \ldots\ (r+1)}.$$

15. $$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}.(a)^{-\frac{1}{2}-3}.\left(\frac{1}{x}\right)^3=\frac{5}{16}a^{-\frac{7}{2}}.\frac{1}{x^3}.$$

16. $$\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2.3.4}.(a^2)^{\frac{3}{2}-4}.(-b^2)^4=\frac{3}{128}a^{-5}.b^8.$$

17. $$-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1.2.3.4.5.6.7.8}.(a^2)^{\frac{1}{2}-8}.(2x^2)^8$$
$$=-\frac{429}{128}.a^{-15}.x^{16}.$$

18. $$-\frac{m.(m+1)\ .\ .\ .\ (m+8)}{1.2\ .\ .\ .\ 9}.a^{-(m+9)}.b^9.$$

19. $$\frac{\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ .\ .\ .\ \left(\frac{1}{m}-5\right)}{1.2\ .\ .\ .\ 6}a^{\frac{1}{m}-6}b^6$$
$$=\frac{(1-m)(1-2m)\ .\ .\ .\ (1-5m)}{1.2\ .\ .\ .\ 6.m^6}.a^{\frac{1}{m}-6}.b^6.$$

CLVI.

1. $$\sqrt[3]{31}=\sqrt[3]{(27+4)}=3\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$$
$$=3\left\{1+\frac{1}{3}\cdot\frac{4}{27}+\frac{1}{2}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\frac{16}{729}\right.$$
$$+\frac{1}{6}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\frac{64}{19683}$$
$$\left.+\frac{1}{24}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\left(\frac{1}{3}-3\right).\frac{256}{531441}+\ .\ .\ .\ .\right\}$$
$$=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\frac{2560}{43046721}+\ .\ .\ .\ .$$
$$=3+\frac{6085724}{43046721}=3{\cdot}1413749\ .\ .\ .\ .$$

2. $\sqrt[7]{108}=\sqrt[7]{(128-20)}=2\left(1-\frac{5}{32}\right)^{\frac{1}{7}}$

$$=2\left\{1-\frac{1}{7}\cdot\frac{5}{32}+\frac{1}{2}\cdot\frac{1}{7}\left(\frac{1}{7}-1\right)\frac{25}{1024}\right.$$

$$-\frac{1}{6}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)\cdot\frac{125}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\cdot\left(\frac{1}{7}-2\right)\cdot\left(\frac{1}{7}-3\right)\cdot\frac{625}{1048576}-\ldots\right\}$$

$$=2\left\{1-\frac{5}{224}-\frac{75}{50176}-\frac{1625}{11239424}-\frac{40625}{2517630976}\ldots\right\}$$

$$=2\left\{1-\frac{56197120+3763200+364000+40625}{2517630976}\right\}$$

$$=2\left(1-\frac{60364945}{2517630976}\right)=\frac{2457266031}{1258815488}=1{\cdot}95204\ldots$$

3. $\sqrt[5]{260}=\sqrt[5]{(243+17)}=3\left(1+\frac{17}{243}\right)^{\frac{1}{5}}$

$$=3\left\{1+\frac{1}{5}\cdot\frac{17}{243}+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{289}{59049}\right.$$

$$\left.+\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{4913}{14348907}\right\}$$

$$=3.\left\{1+\frac{17}{1215}-\frac{578}{1476225}+\frac{29478}{1793613375}\ldots\right\}$$

$$=3\left\{1+\frac{25095825-702270+29478}{1793613375}\ldots\right\}$$

$$=3.\left\{1+\frac{24423033}{1793613375}\right\}=3+\frac{24423033}{597871125}=3{\cdot}04084\ldots$$

4. $\sqrt[5]{31}=\sqrt[5]{(32-1)}=2\left(1-\frac{1}{32}\right)^{\frac{1}{5}}=2\left\{1-\frac{1}{5}\cdot\frac{1}{32}\right.$

$$+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{1}{1024}-\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{1}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\left(\frac{1}{5}-3\right)\cdot\frac{1}{1048576}\cdots\right\}$$

$$=2\left\{1-\frac{1}{160}-\frac{1}{12800}-\frac{3}{4096000}-\frac{21}{655360000}\cdots\right\}$$

$$=2\left\{1-\frac{4096000+51200+480+21}{655360000}\cdots\right\}$$

$$=2\left\{1-\frac{4147701}{655360000}\right\}=1\cdot98734\ldots$$

CLVII.

1.
$$\begin{array}{r} 23561 \\ 42513 \\ 645325 \\ \hline 1045032 \end{array}$$

2.
$$\begin{array}{r} 3074852 \\ 4635628 \\ 1247153 \\ \hline 10070344 \end{array}$$

3.
$$\begin{array}{r} 358423 \\ 267862 \\ \hline 80451 \end{array}$$

4.
$$\begin{array}{r} 211010 \\ 124321 \\ \hline 31134 \end{array}$$

5.
$$\begin{array}{r} 57264 \\ 675 \\ \hline 354604 \\ 513354 \\ 434070 \\ \hline 51117344 \end{array}$$

6.
$$\begin{array}{r} 1456 \\ 6451 \\ \hline 1456 \\ 6523 \\ 11312 \\ 13101 \\ \hline 14332216 \end{array}$$

7.
$$\begin{array}{r} 5)\underline{243012} \\ 31450 \text{ rem. } 2. \end{array}$$

8.
$$\begin{array}{r} 6)\underline{3756025} \\ 522256 \text{ rem. } 1. \end{array}$$

9.
$$\begin{array}{l|l} & 25|40|05|44\,(4112 \\ & \underline{24} \\ 121 & 140 \\ & \underline{121} \\ 1221 & 1505 \\ & \underline{1221} \\ 12222 & 24444 \\ & \underline{24444} \end{array}$$

10.
$$\begin{array}{l|l} & 5|68|98|t1\,(2437 \\ & \underline{4} \\ 44 & 168 \\ & \underline{154} \\ 483 & 1498 \\ & \underline{1209} \\ 4867 & 28et1 \\ & \underline{28et1} \end{array}$$

CLVIII.

1.

7	1828
7	261 − 1
7	37 − 2
7	5 − 2
	0 − 5

2.

6	1820
6	303 − 2
6	50 − 3
6	8 − 2
6	1 − 2
	0 − 1

3.

12	43751
12	3645 − *e*
12	303 − 9
12	25 − 3
12	2 − 1
	0 − 2

4.

5	3700
5	740 − 0
5	148 − 0
5	29 − 3
5	5 − 4
5	1 − 0
	0 − 1

5.

2	7631
2	3815 − 1
2	1907 − 1
2	953 − 1
2	476 − 1
2	238 − 0
2	119 − 0
2	59 − 1
2	29 − 1
2	14 − 1
2	7 − 0
2	3 − 1
2	1 − 1
	0 − 1

6.

12	215855
12	17987 − *e*
12	1498 − *e*
12	124 − *t*
12	10 − 4
	0 − *t*

7.

7	790158
7	112879 − 5
7	16125 − 4
7	2303 − 4
7	329 − 0
7	47 − 0
7	6 − 5
	0 − 6

8.

4	34002
4	4334 − 1
4	1043 − 2
4	122 − 0
4	14 − 1
4	2 − 1
	0 − 2

9.

12	8978
12	816 − 2
12	75 − 1
12	6 − *t*
	0 − 6

10.

12	3256
12	166 − 4
12	11 − 1
	0 − 8

11.

8	37704
8	4311 − 5
8	480 − 1
8	54 − 4
8	6 − 1
	0 − 6

12.

4	5056
4	1165 − 0
4	214 − 3
4	36 − 1
4	6 − 3
4	1 − 2
	0 − 1

13.

7	654321
7	*e*0738 − 5
7	16*e*3*e* − 3
7	2858 − 3
7	478 − 0
7	7*e* − 3
7	11 − 4
7	1 − 6
	0 − 1

14.

11	2304
11	104 − *t*
11	2 − 7
	0 − 2

CLIX.

1. $$\left.\begin{array}{l}\frac{25}{36}\times 6=4+\frac{6}{36}\\[4pt] \frac{6}{36}\times 6=1+0\end{array}\right\}\therefore \cdot 41 \text{ is the result.}$$

2. $\frac{3}{11}\times 7=1+\frac{10}{11}$; $\frac{10}{11}\times 7=6+\frac{4}{11}$; $\frac{4}{11}\times 7=2+\frac{6}{11}$;

$\frac{6}{11}\times 7=3+\frac{9}{11}$; $\frac{9}{11}\times 7=5+\frac{8}{11}$; $\frac{8}{11}\times 7=5+\frac{1}{11}$;

$$\frac{1}{11}\times 7=1+\frac{7}{11};\ \frac{7}{11}\times 7=4+\frac{5}{11};\ \frac{5}{11}\times 7=3+\frac{2}{11};$$

$$\frac{2}{11}\times 7=1+\frac{3}{11};\ \frac{3}{11}\times 7=1+\frac{10}{11}$$

$\therefore$ result is $\cdot\dot{1}6235504\dot{3}$.

3.
$$\begin{array}{r|l} 9 & 23 \\ 9 & 2-5 \\ & 0-2 \end{array} \qquad \begin{array}{r} \cdot 125 \\ 9 \\ \hline 1\cdot 125 \\ 9 \\ \hline 1\cdot 125 \end{array}$$

$\therefore$ result is $25\cdot\dot{1}$.

4.
$$\begin{array}{r|l} 6 & 1820 \\ 6 & 303-2 \\ 6 & 50-3 \\ 6 & 8-2 \\ 6 & 1-2 \\ & 0-1 \end{array} \qquad \begin{array}{r} \cdot 3375 \\ 6 \\ \hline 2\cdot 0250 \\ 6 \\ \hline 0\cdot 1500 \\ 6 \\ \hline 0\cdot 9000 \\ 6 \\ \hline 5\cdot 4000 \\ 6 \\ \hline 2\cdot 4000 \end{array}$$

$\therefore$ result is $12232\cdot 2005\dot{2}$.

5. $2x^5+x^4+2x^3+5x^2+4x+2-17486=0$;

$(x-6)(2x^4+13x^3+80x^2+485x+2914)=0$;

$\therefore x=6$. Hence the scale is senary.

6. $x^6+7x^5+4x^4+6x^3+3x^2+5-511173=0$;

$(x-8)(x^5+15x^4+124x^3+998x^2+7987x+63896)=0$;

$\therefore x=8$. Hence the scale is octenary.

7. (1.) Let N be the number, and suppose

$$N=a.10^n+b.10^{n-1}+ \ldots +m.100+p.10+q.$$

Then $N=a(10^n-1)+b(10^{n-1}-1)+ \ldots +m(100-1)$
$+p(10-1)+(a+b+ \ldots +m+p+q).$

Now all the expressions $10^n-1, 10^{n-1}-1 \ldots 10^2-1, 10-1$ are divisible by $10-1$, or 9, and therefore by 3.

Hence N is divisible by 3, if $a+b+ \ldots +m+p+q$ be divisible by 3.

(2.) Let $N=a.10^n+b.10^{n-1}+ \ldots +m.100+p.10+q.$

Now 100 and all its multiples are divisible by 4,

$\therefore$ N is divisible by 4 if $10p+q$ be divisible by 4.

(3.) Let $N=a.10^n+b.10^{n-1}+ \ldots +m.100+p.10+q.$

Now 1000 and all its multiples are divisible by 8,

$\therefore$ N is divisible by 8 if $100m+10p+q$ be divisible by 8.

(4.) Let $N=a.10^n+b.10^{n-1}+ \ldots +m.100+p.10+q.$

Now 10 and all its multiples are divisible by 5,

$\therefore$ N is divisible by 5 if $q=5$ or $q=0$.

(5.) Let N be the number, $p_n, p_{n-1}, \ldots p_4, p_3, p_2, p_1$, the digits.

Then $N=p_1+10.p_2+100p_3+1000p_4+\ldots+10^{n-1}.p_{n-1}+10^np^n$
$=p_1-p_2+p_3-p_4+ \ldots +(-1)^n.p_n$
$+p_2(10+1)+p_3(10^2-1)+p_4(10^3+1)+\ldots+p_n\{10^n-(-1)^n\}.$

Now $10+1$, $10^2-1 \ldots$ are all divisible by $10+1$, or 11,

$\therefore$ N is divisible by 11, if $(p_1+p_3+ \ldots)-(p_2+p_4 \ldots)$ be divisible by 11.

8. Let $N=a.r^n+b.r^{n-1}+ \ldots +m.r^2+pr+q,$

then $n=a+rb+ \ldots +m.r^{n-2}+p.r^{n-1}+q.r^n.$

Then $N-n=a(r^n-1)+b.(r^{n-1}-r)+ \ldots -m(r^{n-2}-r^2)$
$-p(r^{n-1}-r)-q(r^n-1)$, and each of the factors r^n-1, $r^{n-1}-r \ldots r^{n-2}-r^2, r^{n-1}-r, r^n-1$ is divisible by $r-1$;

$\therefore$ $N-n$ is divisible by $r-1$.

CLX.

1.
$$\begin{array}{r} \overline{3}\cdot1651553 \\ \overline{4}\cdot7505855 \\ 6\cdot6879746 \\ \overline{2}\cdot6150026 \\ \hline \overline{1}\cdot2187180 \end{array}$$

2.
$$\begin{array}{r} \overline{4}\cdot6843785 \\ \overline{5}\cdot6650657 \\ 3\cdot8905196 \\ \overline{3}\cdot4675284 \\ \hline \overline{7}\cdot7074922 \end{array}$$

3.
$$\begin{array}{r} 2\cdot5324716 \\ 3\cdot6650657 \\ \overline{5}\cdot8905196 \\ \cdot3156215 \\ \hline 2\cdot4036784 \end{array}$$

4.
$$\begin{array}{r} 2\cdot483269 \\ \overline{3}\cdot742891 \\ \hline 4\cdot740378 \end{array}$$

5.
$$\begin{array}{r} \overline{2}\cdot352678 \\ \overline{5}\cdot428619 \\ \hline 2\cdot924059 \end{array}$$

6.
$$\begin{array}{r} \overline{5}\cdot349162 \\ \overline{3}\cdot624329 \\ \hline \overline{3}\cdot724833 \end{array}$$

7.
$$\begin{array}{r} \overline{2}\cdot4596721 \\ 3 \\ \hline \overline{5}\cdot3790163 \end{array}$$

8.
$$\begin{array}{r} \overline{7}\cdot429683 \\ 6 \\ \hline \overline{40}\cdot578098 \end{array}$$

9.
$$\begin{array}{r} \overline{9}\cdot2843617 \\ 7 \\ \hline \overline{62}\cdot9905319 \end{array}$$

10.
$$3\,)\,\overline{6}\cdot3725409 \qquad \overline{2}\cdot1241803$$

11.
$$6\,)\,\overline{14}\cdot432962 \qquad \overline{3}\cdot738827$$

12.
$$9\,)\,\overline{4}\cdot53627188 \qquad \overline{1}\cdot61514132$$

CLXI.

1. $\text{Log } 128 = \log 2^7 = 7 \log 2 = 2\cdot1072100.$

$$\text{Log } 125 = \log \frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$$
$$= 3 - 3 \log 2 = 3 - \cdot9030900 = 2\cdot0969100.$$

$$\text{Log } 2500 = \log \frac{10000}{4} = \log 10000 - \log 4 = 4 - 2 \log 2$$
$$= 4 - \cdot6020600 = 3\cdot3979400.$$

2. $\text{Log } 50 = \log \frac{100}{2} = \log 100 - \log 2 = 2 - \cdot3010300 = 1\cdot6989700.$

$$\text{Log } \cdot005 = \log \frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \overline{3}\cdot6989700.$$

$$\text{Log } 196 = \log (49 \times 4) = 2 \log 7 + 2 \log 2 = 2\cdot2922560.$$

3. $\text{Log } 6 = \log 3 + \log 2 = {\cdot}7781513.$

$\text{Log } 27 = 3 \log 3 = 1{\cdot}4313639.$

$\text{Log } 54 = \log (27 \times 2) = 3 \log 3 + \log 2 = 1{\cdot}7323939.$

$\text{Log } 576 = \log (9 \times 64) = 2 \log 3 + 6 \log 2 = 2{\cdot}7604226.$

4. $\text{Log } 60 = \log (2 \times 3 \times 10) = \log 2 + \log 3 + \log 10 = 1{\cdot}7781513.$

$$\text{Log } {\cdot}03 = \log \frac{3}{100} = \log 3 - 2 = {\cdot}4771213 - 2 = \bar{2}{\cdot}4771213.$$

$$\text{Log } 1{\cdot}05 = \log \frac{105}{100} = \log \frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = {\cdot}0211893.$$

$$\text{Log } {\cdot}0000432 = \log \frac{16 \times 27}{10000000} = 4 \log 2 + 3 \log 3 - 7 = \bar{5}{\cdot}6354839.$$

5. $$\text{Log } {\cdot}00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log \left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$$

$$= \frac{1}{2} \{\log 18 - \log 2\} + \log 100 - \log 4 - 5.$$

$$= \frac{1}{2} \{1{\cdot}2552725 - {\cdot}3010300\} + 2 - {\cdot}6020600 - 5$$

$$= {\cdot}4771213 - {\cdot}6020600 - 3 = \bar{4}{\cdot}8750613.$$

$$\text{Log } 31{\cdot}5 = \log (21 \times 3 \times 5) - 1 = \log 21 + \log 3 + 1 - \log 2 - 1.$$

$$= \log 21 + \frac{1}{2} \{\log 18 - \log 2\} - \log 2$$

$$= 1{\cdot}3222193 + {\cdot}4771212 - {\cdot}3010300 = 1{\cdot}4983105.$$

6. $$\text{Log } 2 = \log \frac{10}{5} = 1 - \log 5 = {\cdot}3010300$$

$$\text{Log } {\cdot}064 = \log. \frac{2^6}{1000} = 6 \log 2 - 3 = 6 - 6 \log 5 - 3 = \bar{2}{\cdot}8061800$$

$$\text{Log } \left\{\frac{2^{60}}{5^{20}}\right\}^{\frac{1}{14}} = \frac{1}{14} (60 \log 2 - 20 \log 5)$$

$$= \frac{1}{7} (30 - 30 \log. 5 - 10 \log 5) = \frac{1}{7} (30 - 27{\cdot}9588000)$$

$$= \frac{1}{7} (2{\cdot}0412000) = {\cdot}2916000.$$

7. $\text{Log } 5 = \log \frac{10}{2} = 1 - \cdot 3010300 = \cdot 6989700$

$$\log \cdot 125 = \log \frac{5^3}{1000} = 3 \log 5 - 3 = 2 \cdot 0969100 - 3 = \bar{1} \cdot 0969100$$

$$\text{Log} \left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}} = \log 5^{\frac{90}{15}} - \log 2^{\frac{40}{15}} = \log 5^6 - \log 2^{\frac{8}{3}}$$

$$= 6 \log 5 - \frac{8}{3} \log 2 = 6 \ (\log 10 - \log 2) - \frac{8}{3} \log 2$$

$$= 4 \cdot 1938200 - \cdot 8027467 = 3 \cdot 3910733.$$

8. $-2, 0, 2 ; 1, 0, -1.$

9. 1593 is greater than 10^3 and less than 10^4; characteristic 3; 1593 is greater than 12^2 and less than 12^3; characteristic 2.

10. $\frac{4^{3y}}{2^{4y}} = 8 ; \frac{2^{6y}}{2^{4y}} = 2^3 ; 2^{2y} = 2^3 ; 2y = 3$, etc.

11. (*a*) $\log 2 = \frac{1}{2} \log 4 = \cdot 3010300.$

$$\text{Log } 25 = \log 100 - \log 4 = 2 - \cdot 6020600 = 1 \cdot 3979400$$

$$\text{Log } 83 \cdot 2 = \log (80 \times 1 \cdot 04) = \frac{3}{2} \log 4 + \log 10 + \log 1 \cdot 04$$

$$= \cdot 9030900 + 1 + \cdot 0170333 = 1 \cdot 9201233.$$

$$\text{Log } (\cdot 625)^{\frac{1}{100}} = \frac{1}{100} \left\{ \log 625 - \log 1000 \right\} = \frac{1}{100} \left\{ 2 \log 25 - 3 \right\}$$

$$= \frac{1}{100} \left\{ 2 \log 100 - 2 \log 4 - 3 \right\} = \frac{1}{100} \left\{ 4 - 1 \cdot 2041200 - 3 \right\}$$

$$= - \cdot 0020412 = \bar{1} \cdot 9979588.$$

(*b*) $\text{Log } (1 \cdot 04)^{6000} = 6000 \log 1 \cdot 04 = 6000 \times \cdot 0170333$

$= 102 \cdot 1998000$; $\therefore$ number of digits is 103.

12. (*a*) $\text{Log } 5 = \frac{1}{2} \log 25 = \cdot 6989700$

$\text{Log } 4 = 2 - \log 25 = \cdot 6020600$

$\text{Log } 51 \cdot 5 = \log 5 + \log 10 \cdot 3 = \cdot 6989700 + 1 \cdot 0128372 = 1 \cdot 7118072$

$\text{Log } (\cdot 064)^{\frac{1}{100}} = \frac{1}{100} \left\{ \log 64 - \log 1000 \right\} = \frac{1}{100} \left\{ 3 \log 4 - 3 \right\}$

$= \frac{1}{100} \left\{ 1 \cdot 8061800 - 3 \right\} = - \cdot 0119382 = \bar{1} \cdot 9880618.$

(*b*) $\text{Log } (1 \cdot 03)^{600} = 600 \log 1 \cdot 03 = 600 \times \cdot 0128372$

$= 7 \cdot 7023200$; $\therefore$ number of digits is 8.

13. $\text{Log } 7623 = 2 \log 3 + 2 \log 11 + \log 7$

$= \cdot 9542426 + 2 \cdot 0827854 + \cdot 8450980 = 3 \cdot 8821260$

$\text{Log } \frac{77}{300} = \log 7 + \log 11 - \log 3 - \log 100$

$= \cdot 8450980 + 1 \cdot 0413927 - \cdot 4771213 - 2 = \bar{1} \cdot 4093694$

$\text{Log } \frac{3}{539} = \log 3 - \log 11 - 2 \log 7$

$= \cdot 4771213 - 1 \cdot 0413927 - 1 \cdot 6901960 = \bar{3} \cdot 7455326.$

14. (1.) $x \log 4096 = \log 8 - x \log 64$; $4x \log 8 = \log 8 - 2x \log 8$;

$4x = 1 - 2x$; $6x = 1$; $x = \frac{1}{6}$.

(2.) $(2 \cdot 5)^x = 6 \cdot 25 = (2 \cdot 5)^2$; $\therefore x = 2.$

(3.) $(ab)^x = m$; $x \log (ab) = \log m$

$\therefore x = \frac{\log m}{\log a + \log b}.$

(4.) $x(m \log a + 2 \log b) = \log c$, etc.

(5.) $3x \log a + (4 - x) \log b = (2x - 1) \log c$

$x(3 \log a - \log b - 2 \log c) = -4 \log b - \log c$, etc.

(6.) $x (\log a + m \log b) = \log c - 3x \log c$

$x (\log a + m \log b + 3 \log c) = \log c$, etc.

CLXII.

1. $P(1+r)^n=2P$; $(1+r)^n=2$; $\left(1+\frac{4}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 104-\log 100}=\frac{\cdot 3010300}{\cdot 0170333}=17\cdot 6\ \ldots$$

2. $P(1+r)^n=2P$; $\left(1+\frac{3}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 4\ \ldots$$

3. $\left(1+\frac{10}{100}\right)^n=2$; $n=\frac{\log 2}{\log 1\cdot 1}=\frac{\cdot 3010300}{\cdot 0413927}=7\cdot 2725\ \ldots$

4. $\left(1+\frac{5}{100}\right)^n=3$; $n=\frac{\log 3}{\log 1\cdot 05}=\frac{\cdot 4771213}{\cdot 0211893}=22\cdot 5$ nearly.

5. $P(1+r)^n=2P$; $\therefore n=\frac{\log 2}{\log(1+r)}$

$$P(1+2r)^m=2P\ ;\ \therefore m=\frac{\log 2}{\log(1+2r)}$$

$$\therefore \frac{m}{n}=\frac{\log(1+r)}{\log(1+2r)}.$$

Now $1+2r$ is less than $(1+r)^2$

$$\therefore \frac{m}{n} \text{ is greater than } \frac{\log(1+r)}{\log(1+r)^2}, \text{ or, } \frac{\log(1+r)}{2\log(1+r)}, \text{ or, } \frac{1}{2}.$$

6. $1000(1+r)^n=1800$; $\left(1+\frac{5}{100}\right)^n=1\cdot 8$

$$\therefore n\ \log\ 1\cdot 05=\log\ 1\cdot 8\ ;\ n=\frac{\cdot 2552725}{\cdot 0211893}=12 \text{ nearly.}$$

7. $P(1+r)^{2n}=2P$; $\left(1+\frac{3}{100}\right)^{2n}=2$

$$\therefore 2n=\frac{\log 2}{\log 1\cdot 03}=\frac{\cdot 3010300}{\cdot 0128372}=23\cdot 449\ \ldots$$

$$\therefore n=11\cdot 724\ \ldots.$$

FINIS.

3, WATERLOO PLACE, PALL MALL.
July, 1879.

Educational Works

PUBLISHED BY

MESSRS. RIVINGTON

NEW BOOKS in Preparation and in the Press

THE ANTIQUITIES OF GREECE

Translated from the German of G. F. SCHOEMANN,

By E. G. HARDY, M.A., *late Fellow of Jesus College, Oxford;*
and
J. S. MANN, M.A., *sometime Scholar of Exeter College, Oxford.*

2 Vols. 8vo. [*Vol. I. just ready.*

This Translation is intended to make known the best and most recent results of German research on the whole public and private life of the Greeks. No English book on the subject is at once so exhaustive and searching. The original is distinguished by its interesting style, continuous treatment, and judicious arrangement; it not only describes Greek institutions, but traces their rise historically, and explains them in the light of Greek ideas and theories. The book will, it is hoped, become a standard authority for English, as it is for German, Scholars. It is peculiarly adapted for use at the Universities, and also for the Upper Forms at Schools, while it should not be unwelcome to the general literary public.

VOLUME I. contains (1) Homeric Greece, (2) Historical Greece. Historical Greece is divided into the three following divisions :—

PART I.—GENERAL CHARACTERISTICS OF THE GREEK STATE.

Difference of race—Idea and conditions of the State—Chief forms of constitution—The citizens and the labouring classes—System of public discipline, &c., &c.

PART II. — HISTORICAL STATEMENTS WITH RESPECT TO THE CONSTITUTIONS OF PARTICULAR STATES.

Monarchy — Oligarchy — Lawgivers — Tyrants — Democracy — Theoretical Reformers—Party struggles, &c., &c.

PART III.—DESCRIPTION OF PRINCIPAL STATES IN DETAIL.

Political, social, military, and financial condition of Sparta, Crete, and Athens. SPARTA: Spartiatæ — Helots—Legislation of Lycurgus—Military system—Hellenic policy, &c. ATHENS: Land and People — Solonian legislation — Development of Democracy—Slaves—Financial system—Judicial system, &c.

VOLUME II. will contain: PART I.—INTERNATIONAL RELATIONS.
PART II.—GREEK RELIGION.

New Books in Preparation and in the Press—*continued.*

A SECOND LATIN WRITER.

By GEORGE L. BENNETT, M.A., *Head-Master of the High School, Plymouth, late Assistant-Master at Rugby School.*

This work, in continuation of the First Latin Writer, will give hints on writing Latin Prose for boys about to commence the rendering of continuous passages from English Authors into Latin. There will be a large collection of Exercises, graduated according to their difficulty, with Notes. See page 18.

A FIRST GREEK WRITER.

By ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.* See page 28.

HOMER'S ILIAD.

Edited, with Notes at the end for the Use of Junior Students, by ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.*

Small 8vo. See page 34.

Books III. and IV.

CAESAR. DE BELLO GALLICO. BOOKS I. to III.

Also Book I. separately.

Edited by J. H. MERRYWEATHER, M.A., *and* C. C. TANCOCK, M.A., *Assistant-Masters at Charterhouse School.*

Small 8vo. See page 19. [*Just ready.*

A PRIMER OF GREEK SYNTAX.

By the Editors of the "Primer of Greek Accidence." See page 26.

HISTORY OF THE ROMANS TO THE ESTABLISHMENT OF IMPERIALISM.

By J. S. REID, M.L., *Classical Lecturer at Christ's College, Cambridge.*

This work is intended to be used by the higher Forms in Public Schools, and by Junior Students in the Universities. It aims at exhibiting in outline the growth of the Roman national life in all departments. Military history will not be neglected, but attention will be particularly directed towards the political and social changes, and the development of law, literature, religion, art, science, and social life. Care will be taken to bring the whole narrative into accord with the present state of knowledge, and also to present the facts of Roman History in a form likely to interest the Students for whom the work is intended. See page 9.

New Books in Preparation and in the Press—*continued.*

A Practical Greek Method for Beginners.

Being a graduated application of Grammar to Translation and Composition. By F. Ritchie, M.A., *and* E. H. Moore, M.A., *Assistant-Masters at the High School, Plymouth.*

See page 26. [*Part I. nearly ready.*

The aim of this book, which is at once a Grammar and Exercise Book, is to secure an uniform method of teaching Grammar, and to afford abundant practice in inflexions, &c., at the time the Grammar is being learned.

Part I. contains the Substantives, Adjectives, Pronouns, and Regular Pure Verbs with Exercises (English-Greek and Greek-English), introducing the main rules of Syntax of the Simple Sentence.

Part II. contains the remaining Classes of Regular and Irregular Verbs, with consecutive pieces for Translation, introducing progressively the Syntax of the Compound Sentence.

A Short History of England for Schools.

By F. York-Powell, M.A., *Lecturer at Christ Church, and Public Examiner, Oxford.* With Maps and Illustrations.

See page 12. [*In the Press.*

This book is intended to fill a place which is not precisely taken by any existing School History. Its aim will be to give a brief and correct, but as far as possible interesting, account of the main points of importance and leading characters of each reign, as well as to convey some notion of the chief social and domestic features of the principal epochs, &c., and it is intended for the use of scholars in the middle and upper forms of schools who, while they have outgrown the stage at which Primers are useful, are not yet able to cope with larger and more complete Histories. The want of this class of book has been long felt by teachers and examiners; and while engaged in the preparation of the work, the Author has had the benefit of practical advice from several schoolmasters who were anxious for the appearance of such a book as he is here attempting to supply.

While working on the lines of Mr. Bright's excellent History of England, to which this is indeed in some sort an Introduction, the Author has of course gone over the ground afresh, writing wherever he was able, from the original authorities; he has also by the aid of typical illustrations, maps, plans, tables, &c., endeavoured as far as he could to appeal to the eye on behalf of a subject which should above all others be vividly pictured by learners if they are ever to remember or understand it.

A Chronological Handbook of English History.

A Framework for use in Class-teaching and Lecturing.

By Arthur H. D. Acland, M.A., *Christ Church, Oxford, an* Cyril Ransome, M.A., *Merton College, Oxford; Professor of Modern Literature and History, Yorkshire College, Leeds.*

See page 11. [*In the Press.*

New Books in Preparation and in the Press—*continued.*

XENOPHON'S AGESILAUS.

Edited by R. W. TAYLOR, M.A., *Head-Master of Kelly College, Tavistock.*

THE MERCHANT OF VENICE.

Edited by the same. Forming one of the volumes of the "Rugby Plays of Shakspere." See page 6.

SCOTT'S ROKEBY.

Edited by the same. Forming one of the volumes of the "English School-Classics."

ARNOLD'S PRACTICAL INTRODUCTION TO GREEK PROSE COMPOSITION.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford. A New and Revised Edition.* See page 32.

THE POLITICAL SPEECHES OF DEMOSTHENES.

Translated into English, with Introductions and Appendices.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

AN EASY GREEK READING BOOK.

By C. HEPWORTH GIBSON, M.A., *Assistant-Master at Merchant Taylors' School.* See page 35.

THE ELEMENTS OF PHYSICAL GEOGRAPHY.

By H. G. SEELEY, F.R.S., *Professor of Geography at King's College, London.*

ROMEO AND JULIET.

Edited by Rev. CHARLES E. MOBERLY, M.A., *Assistant-Master at Rugby School. Forming one of the volumes of the "Rugby Plays of Shakspere."* See page 6.

OLIVER CROMWELL.

By F. W. CORNISH, M.A., *Assistant-Master at Eton College. Forming one of the volumes of "Historical Biographies."* See page 11.

THE ROMAN REVOLUTION. From B.C. 133 to the BATTLE OF ACTIUM.

By HENRY F. PELHAM, M.A., *Fellow of Exeter College, Oxford.*

KEYS

KEYS marked **A** are published to the following Educational Works *for the use of Tutors only.* They can *only* be obtained by direct application to the Publishers, who will send a printed Form, to be filled up by the Tutor requiring the KEY. They cannot be supplied through Booksellers.

A

		Nett Price of the Key. s.	d.
BENNETT'S	First Latin Writer	5	0
———	Easy Latin Stories for Beginners	5	0
GEPP'S	Arnold's Henry's First Latin Book	5	0
———	Exercises in Latin Elegiac Verse	5	0
SIDGWICK'S	Greek Prose Composition	5	0
ARNOLD'S	Henry's First Latin Book	1	0
———	Supplementary Exercises	1	6
———	Second Latin Book	2	0
———	First Verse Book	1	0
———	Latin Verse Composition	2	0
———	Longer Latin Exercises. Part II.	2	6
———	Latin Prose Composition. Part I.	1	6
———	First Greek Book	1	6
———	Greek Prose Composition. Part I.	1	6
SARGENT AND DALLIN'S	Materials and Models for Latin Prose Composition. Latin Version. Sixty Selected Pieces.	5	0
SARGENT AND DALLIN'S	Materials and Models for Greek Prose Composition. Greek Version. Selected Pieces.	7	6

B

Keys to the following are sold to the Public without restriction:

		s.	d.
ARNOLD'S	First German Book	2	6
———	Second German Book	1	0
———	First French Book	2	6
———	First Italian Book	1	6
———	First Hebrew Book	3	6
SMITH'S (J. HAMBLIN)	Elementary Algebra	9	0
———	Arithmetic	9	0
———	Statics and Hydrostatics	6	0
———	Trigonometry	7	6
———	Latin Prose Exercises	5	0

The above-mentioned Keys are published at NETT prices, and no reduction is made to Tutors.

ENGLISH

SELECT PLAYS OF SHAKSPERE

RUGBY EDITION

With Introduction and Notes to each Play.
Small 8vo.

AS YOU LIKE IT. *2s.*

MACBETH. *2s.*

HAMLET. *2s. 6d.*

KING LEAR. *2s. 6d.*
With Notes at the end of the Volume.

ROMEO AND JULIET. [*In preparation.*

Edited by the Rev. CHARLES E. MOBERLY, M.A., *Assistant-Master at Rugby School, and formerly Scholar of Balliol College, Oxford.*

CORIOLANUS. *2s. 6d.*

Edited by ROBERT WHITELAW, M.A., *Assistant-Master at Rugby School, formerly Fellow of Trinity College, Cambridge.*

THE TEMPEST. *2s.*

Edited by J. SURTEES PHILLPOTTS, M.A., *Head-Master of Bedford Grammar School, formerly Fellow of New College, Oxford.*
With Notes at the end of the Volume.

THE MERCHANT OF VENICE.

Edited by R. W. TAYLOR, M.A., *Head-Master of Kelly College, Tavistock, and late Fellow of St. John's College, Cambridge.*
With Notes at the end of the Volume.
[*In preparation.*

The Rudiments of English Grammar and Composition.

By J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer at St. Peter's College, Cambridge.*

Crown 8vo. 2*s*. 6*d*.

"Though prepared specially for the requirements of the University of Cambridge in the local examinations, this grammar is well worthy of the attention of all who are engaged in the teaching of English."

Glasgow Herald.

"The book is simply and intelligently written; it contains nearly all that a young student should know, and it is full of well chosen examples from English classics to illustrate the different subjects treated."

Spectator.

"This work is very elementary, but, like all Mr. Hamblin Smith's useful books, wonderfully lucid. It should well fulfil its design as a text-book for the local examinations. We are pleased to note the varied and tasteful selection of the illustrative quotations." — *Oxford and Cambridge Undergraduates' Journal.*

"We have here the most difficult study in the English language and what is usually the dryest reduced to language so simple that the smallest child, even before it learns to read can understand the parts of speech of the tongue it articulates, and rendered so interesting as entirely to remove the stigma of dullness."

Cambridge Chronicle.

The Beginner's Drill-book of English Grammar.

Adapted for Middle Class and Elementary Schools.

By JAMES BURTON, T.C.D., *First English Master in the High School of the Liverpool Institute.*

Small 8vo. 1*s*. 6*d*.

Contents:—I. ANALYSIS OF SIMPLE SENTENCES:—On the Simple Sentence—Classification of Words—Change of Words from one Class to another—Order of Words and Breviation of Sentences—Order of Proceeding in Analysing Simple Sentences. II. INFLEXION:—Use of Inflexion and Compounding—Inflexion of the Noun—Inflexion of the Pronoun—Inflexion of the Verb—Tables of Verbs—Inflexion of the Adjective and Adverb. III. SYNTAX: ANALYSIS OF COMPLEX SENTENCES:—Elementary Rules and Cautions—Indirect Objects—The Complex Sentence with the Indicative Mood—The Complex Sentence with the Subjunctive Mood. IV. THE ALPHABET:—Analysis of English Speech-sounds—Insufficiency of the English Alphabet. APPENDIX:—Examples of Analysis and Parsing—Exercises—Examples of Additional Exercises.

Short Readings in English Poetry.

Edited by H. A. HERTZ.

Small 8vo. [*In the Press.*

ENGLISH SCHOOL-CLASSICS

With Introductions, and Notes at the end of each Book.

Edited by FRANCIS STORR, M.A.,

CHIEF MASTER OF MODERN SUBJECTS AT MERCHANT TAYLORS' SCHOOL, LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE, AND BELL UNIVERSITY SCHOLAR.

Small 8vo.

THOMSON'S SEASONS: Winter.
With Introduction to the Series, by the Rev. J. FRANCK BRIGHT, M.A., Fellow of University College, and Historical Lecturer at Balliol, New, and University Colleges, Oxford; late Master of the Modern School at Marlborough College. *1s.*

COWPER'S TASK.
By FRANCIS STORR, M.A., Chief Master of Modern Subjects at Merchant Taylors' School. *2s.*

Part I. (Book I.—The Sofa; Book II.—The Timepiece) *9d.* Part II. (Book III. —The Garden; Book IV.—The Winter Evening) *9d.* Part III. (Book V.—The Winter Morning Walk; Book VI.—The Winter Walk at Noon) *9d.*

COWPER'S SIMPLE POEMS.
By FRANCIS STORR, M.A., Chief Master of Modern Subjects at Merchant Taylors' School.

SCOTT'S LAY OF THE LAST MINSTREL.
By J. SURTEES PHILLPOTTS, M.A., Head Master of Bedford School, formerly Fellow of New College, Oxford. *2s. 6d.*

Part I. (Canto I., with Introduction, &c.) *9d.* Part II. (Cantos II. and III.) *9d.* Part III. (Cantos IV. and V.) *9d.* Part IV. (Canto VI.) *9d.*

SCOTT'S LADY OF THE LAKE.
By R. W. TAYLOR, M.A., Head-Master of Kelly College, Tavistock; late Fellow of St. John's College, Cambridge. *2s.*

Part I. (Cantos I. and II.) *9d.* Part II. (Cantos III. and IV.) *9d.* Part III. (Cantos V. and VI.) *9d.*

NOTES TO SCOTT'S WAVERLEY.
By H. W. EVE, M.A., Head-Master of University College School, London. *1s.*, or with the Text, *2s. 6d.*

TWENTY OF BACON'S ESSAYS.
By FRANCIS STORR, M.A., Chief Master of Modern Subjects at Merchant Taylors' School. *1s.*

SIMPLE POEMS.
Edited by W. E. MULLINS, M.A., Assistant-Master at Marlborough College. *8d.*

SELECTIONS FROM WORDSWORTH'S POEMS.
By H. H. TURNER, B.A., late Scholar of Trinity College, Cambridge. *1s.*

WORDSWORTH'S EXCURSION: The Wanderer.
By H. H. TURNER, B.A., late Scholar of Trinity College, Cambridge. *1s.*

MILTON'S PARADISE LOST.
By FRANCIS STORR, M.A., Chief Master of Modern Subjects at Merchant Taylors' School.
Book I. *9d.* Book II. *9d.*

MILTON'S L'ALLEGRO, IL PENSEROSO, AND LYCIDAS.
By EDWARD STORR, M.A., late Scholar of New College, Oxford. *1s.*

ENGLISH SCHOOL-CLASSICS—continued.

SELECTIONS FROM THE SPECTATOR.
By Osmund Airy, M.A., Assistant-Master at Wellington College. *1s.*

BROWNE'S RELIGIO MEDICI.
By W. P. Smith, M.A., Assistant-Master at Winchester College. *1s.*

GOLDSMITH'S TRAVELLER AND DESERTED VILLAGE.
By C. Sankey, M.A., Head-Master of Bury St. Edmund's Grammar School. *1s.*

EXTRACTS FROM GOLDSMITH'S VICAR OF WAKEFIELD.
By C. Sankey, M.A., Head-Master of Bury St. Edmund's Grammar School. *1s.*

POEMS SELECTED FROM THE WORKS OF ROBERT BURNS.
By A. M. Bell, M.A., Balliol College, Oxford. *2s.*

MACAULAY'S ESSAYS.
MOORE'S LIFE OF BYRON. By Francis Storr, M.A. *9d.*
BOSWELL'S LIFE OF JOHNSON. By Francis Storr, M.A. *9d.*
HALLAM'S CONSTITUTIONAL HISTORY. By H. F. Boyd, late Scholar of Brasenose College, Oxford. *1s.*

SOUTHEY'S LIFE OF NELSON.
By W. E. Mullins, M.A., Assistant-Master at Marlborough College. *2s. 6d.*

GRAY'S POEMS, JOHNSON'S LIFE, AND SELECTION FROM GRAY'S LETTERS.
Edited by Francis Storr, M.A., Chief Master of Modern Subjects at Merchant Taylors' School. *1s.*

*** *The General Introduction to Series will be found in* Thomson's Winter.

HISTORY

History of the Romans to the Establishment of Imperialism.

By J. S. Reid, M.L., *Classical Lecturer at Christ's College, Cambridge.*

This work is intended to be used by the higher Forms in Public Schools, and by Junior Students in the Universities. It aims at exhibiting in outline the growth of the Roman national life in all departments. Military history will not be neglected, but attention will be particularly directed towards the political and social changes, and the development of law, literature, religion, art, science, and social life. Care will be taken to bring the whole narrative into accord with the present state of knowledge, and also to present the facts of Roman History in a form likely to interest the Students for whom the work is intended. [*In preparation.*

HISTORICAL HANDBOOKS

Edited by

OSCAR BROWNING, M.A.,

FELLOW OF KING'S COLLEGE, CAMBRIDGE.

Crown 8vo.

English History in the XIVth Century.

By Charles H. Pearson, M.A., *Head-Master of the Presbyterian Ladies' College, Melbourne, late Fellow of Oriel College, Oxford.*

3*s*. 6*d*.

The Reign of Lewis XI.

By P. F. Willert, M.A., *Fellow of Exeter College, Oxford.*

With Map. 3*s*. 6*d*.

The Roman Empire. A.D. 395–800.

By A. M. Curteis, M.A., *Assistant-Master at Sherborne School, late Fellow of Trinity College, Oxford.*

With Maps. 3*s*. 6*d*.

History of the English Institutions.

By Philip V. Smith, M.A., *Barrister-at-Law; Fellow of King's College, Cambridge.*

Second Edition. 3*s*. 6*d*.

History of Modern English Law.

By Sir Roland Knyvet Wilson, Bart., M.A., *Barrister-at-Law; late Fellow of King's College, Cambridge.*

3*s*. 6*d*.

History of French Literature.

Adapted from the French of M. Demogeot, *by* C. Bridge.

3*s*. 6*d*.

The Age of Chatham and Pitt.

By Sir W. R. Anson, Bart., M.A., *Fellow of All Souls' College, Oxford.* [*In Preparation.*

HISTORICAL BIOGRAPHIES

Edited by

THE REV. M. CREIGHTON, M.A.,

LATE FELLOW AND TUTOR OF MERTON COLLEGE, OXFORD.

With Maps and Plans. Small 8vo.

Now ready.

SIMON DE MONTFORT. *By* M. CREIGHTON, M.A. 2*s.* 6*d.*

THE BLACK PRINCE. *By* LOUISE CREIGHTON. 2*s.* 6*d.*

SIR WALTER RALEGH. *By* LOUISE CREIGHTON. With Portrait. 3*s.*

THE DUKE OF WELLINGTON. *By* ROSAMOND WAITE. With Portrait. 3*s.* 6*d.*

THE DUKE OF MARLBOROUGH. *By* LOUISE CREIGHTON. With Portrait. 3*s.* 6*d.*

OLIVER CROMWELL. *By* F. W. CORNISH, M.A. [*In preparation.*

History of the Church under the Roman Empire, A.D. 30–476.

By the Rev. A. D. CRAKE, B.A., *Chaplain of All Saints' School, Bloxham.*

Second Edition, revised. Crown 8vo. 7*s.* 6*d.*

A Chronological Handbook of English History. A Framework for use in Class-teaching and Lecturing.

By ARTHUR H. D. ACLAND, M.A., *Christ Church, Oxford; and* CYRIL RANSOME, M.A., *Professor of Modern Literature and History, Yorkshire College, Leeds.* [*In the Press.*

A History of England for Children.

By GEORGE DAVYS, D.D., *formerly Bishop of Peterborough.*

New Edition. 18mo. 1*s.* 6*d.*

With numerous Maps and Plans. Second Edition, Revised. Crown 8vo.

A History of England.

By the Rev. J. FRANCK BRIGHT, M.A., *Fellow of University College, and Historical Lecturer at Balliol, New, and University Colleges, Oxford; late Master of the Modern School at Marlborough College.*

Period I.—MEDIÆVAL MONARCHY: The departure of the Romans, to Richard III. From A.D. 449 to A.D. 1485. Second Edition, Revised. 4*s.* 6*d.*

Period II.—PERSONAL MONARCHY: Henry VII. to James II. From A.D. 1485 to A.D. 1688. Second Edition, Revised. 5*s.*

Period III.—CONSTITUTIONAL MONARCHY: William and Mary, to the present time. From A.D. 1689 to A.D. 1837. Second Edition, Revised. 7*s.* 6*d.*

(*See Specimen Page opposite.*)

A Short History of England for Schools.

By F. YORK-POWELL, M.A., *Lecturer at Christ Church, and Public Examiner, Oxford.* With Maps and Illustrations.

[*In the Press.*

This book is intended to fill a place which is not precisely taken by any existing School History. Its aim will be to give a brief and correct, but as far as possible interesting, account of the main points of importance and leading characters of each reign, as well as to convey some notion of the chief social and domestic features of the principal epochs, &c., and it is intended for the use of scholars in the middle and upper forms of schools who, while they have outgrown the stage at which Primers are useful, are not yet able to cope with larger and more complete Histories. The want of this class of book has been long felt by teachers and examiners; and while engaged in the preparation of the work, the Author has had the benefit of practical advice from several schoolmasters who were anxious for the appearance of such a book as he is here attempting to supply.

While working on the lines of Mr. Bright's excellent History of England, to which this is indeed in some sort an Introduction, the Author has of course gone over the ground afresh, writing wherever he was able, from the original authorities; he has also by the aid of typical illustrations, maps, plans, tables, &c., endeavoured as far as he could to appeal to the eye on behalf of a subject which should above all others be vividly pictured by learners if they are ever to remember or understand it.

state of things was for the moment crossed by the death of Alexander (Dec. 1, 1825). The view which his successor Nicholas would take became in the last degree important; Canning, with great wisdom, chose Wellington—opposed indeed to his policy, but personally acceptable to the Russian Czar—as his special ambassador to take the royal congratulations upon the new Emperor's accession, and to continue the negotiations if possible. The appointment met with universal approbation; even Metternich believed that in the hands of Wellington the question must be settled in accordance with his views. It was with much surprise and anger that the Turks and Austrians heard that, on the 4th of April, an arrangement had been arrived at between the Courts of England and Russia. Taking advantage of the very moderate claims of the Greeks, who demanded no more than to be placed on the same footing as the Danubian Principalities, remaining as self-governing but dependent vassals of the Turkish Government, the English minister had succeeded in procuring the signature of a protocol embodying a plan for peaceful intervention.

Protocol between England and Russia. April 1826.

The cause of Greek independence had already excited enthusiasm in England, many volunteers had joined the armies, and money had been subscribed for them. In this enthusiasm Canning in his heart fully joined; from early youth one of his favourite dreams had been the independence of that race to which as an ardent lover of the classics he felt he owed so much. But, true to his principles, and determined to maintain the strict neutrality of England, he had done his best to check any active assistance to the insurgents. According to his view it was necessary that England should intervene with clean hands, and as the friend of both parties. He was also in constant dread of the watchfulness of his Tory enemies, fearing lest any sign of too great favour to Russia should enable them entirely to thwart his plans. Nevertheless the knowledge of the approaching intervention gave a great impetus to the feeling in favour of Greece in England, and men and money were poured in considerable quantities into the peninsula. Lord Cochrane, the most dashing and adventurous of English sailors, had joined the insurgents with an American frigate, General Churchill took command of their armies, yet their destruction seemed immi-

Enthusiasm for Greek independence in England.

MATHEMATICS

RIVINGTONS' MATHEMATICAL SERIES

The following Schools, amongst many others, use this Series:—Eton: Harrow: Rugby: Winchester: Charterhouse: Marlborough: Shrewsbury: Cheltenham: Clifton: City of London School: Haileybury: Tonbridge: Fettes College, Edinburgh: H.M.'s Dockyard Schools, Sheerness and Devonport: Hurstpierpoint: King William's College, Isle of Man: Bradfield College, Reading: St. Peter's, Clifton, York: Birmingham: Bedford: Felsted: Christ's College, Finchley: Liverpool College: Windermere College: Eastbourne College: Competitive College, Bath: Brentwood: Perse School, Cambridge: Queen's College, Cork. Also in use in the Royal Naval College, Greenwich: H.M. Training Ships: the Owen's College, Manchester: Harvard College, U.S.: the Grammar and High Schools of Canada: Melbourne University, Australia: the other Colonies: and some of the Government Schools in India.

OPINIONS OF TUTORS AND SCHOOLMASTERS.

"A person who carefully studies these books will have a thorough and accurate knowledge of the subjects on which they treat."—*H. A. Morgan, M.A., Tutor of Jesus College, Cambridge.*

"We have for some time used your Mathematical books in our Lecture Room, and find them well arranged, and well calculated to clear up the difficulties of the subjects. The examples also are numerous and well-selected."—*N. M. Ferrers, M.A., Fellow and Tutor of Gonville and Caius College, Cambridge.*

"I have used in my Lecture Room Mr. Hamblin Smith's text-books with very great advantage."—*James Porter, M.A., Master of St. Peter's College, Cambridge.*

"For beginners there could be no better books, as I have found when examining different schools."—*A. W. W. Steel, M.A., Fellow and Assistant-Tutor of Gonville and Caius College, Cambridge.*

"I consider Mr. Hamblin Smith's Mathematical Works to be a very valuable series for beginners. His Algebra in particular I think is the best book of its kind for schools and for the ordinary course at Cambridge."
F. Pattrick, M.A., Fellow and Tutor of Magdalen College, Cambridge.

"The series is a model of clearness and insight into possible difficulties."—*Rev. J. F. Blake, St. Peter's College, Clifton, York.*

"I can say with pleasure that I have used your books extensively in my work at Haileybury, and have found them on the whole well adapted for boys."—*Thomas Pitts, M.A., Assistant Mathematical Master at Haileybury College.*

"I can strongly recommend them all."
W. Henry, M.A., Sub-Warden, Trinity College, Glenalmond.

"I consider Mr. Smith has supplied a great want, and cannot but think that his works must command extensive use in good schools."—*J. Henry, B.A., Head-Master, H.M. Dockyard School, Sheerness, and Instructor of Engineers, R.N.*

"We have used your Algebra and Trigonometry extensively at this School from the time they were first published, and I thoroughly agree with every mathematical teacher I have met, that, as school text-books, they have no equals. We are introducing your Euclid gradually into the School."—*Rev. B. Edwardes, sen., Mathematical Master at the College, Hurstpierpoint, Sussex.*

"I consider them to be the best books of their kind on the subject which I have yet seen."—*Joshua Jones, D.C.L., Head-Master, King William's College, Isle of Man.*

"I have very great pleasure in expressing an opinion as to the value of these books. I have used them under very different circumstances, and have always been satisfied with the results obtained."—*C. H. W. Biggs, Editor of the 'Educational Times,' and the 'Monthly Journal of Education.'*

RIVINGTONS' MATHEMATICAL SERIES

ELEMENTARY ALGEBRA. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*

Small 8vo. 3*s.* Without Answers, 2*s.* 6*d.* A KEY. Crown 8vo. 9*s.*

EXERCISES ON ALGEBRA. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 2*s.* 6*d.* (Copies may be had without the Answers.)

ALGEBRA. Part II. *By* E. J. GROSS, M.A., *Fellow of Gonville and Caius Coeglle, Cambridge, and Secretary to the Oxford and Cambridge Schools Examination Board.*

Crown 8vo. 8*s.* 6*d.*

KINEMATICS AND KINETICS. *By* E. J. GROSS, M.A.

Crown 8vo. 5*s.* 6*d.*

GEOMETRICAL CONIC SECTIONS.

By G. RICHARDSON, M.A., *Assistant-Master at Winchester College, and late Fellow of St. John's College, Cambridge.*

Crown 8vo. 4*s.* 6*d.*

A TREATISE ON ARITHMETIC. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 3*s.* 6*d.* (Copies may be had without the Answers.)

A KEY. Crown 8vo. 9*s.*

EXAMINATION FOR TEACHERS' CERTIFICATES: SUGGESTIONS FROM DR. MCLELLAN.—"I therefore recommend all intending candidates to read carefully Hamblin Smith's Arithmetic. Some of this Author's works are already well and favourably known in Ontario; the Arithmetic is one of the most valuable of the series. It explains and illustrates the Unitary Method, showing how the elementary principles of pure science, without being disguised in multitudinous perplexing 'Rules,' can be applied to the solution of all classes of questions.

"If then candidates master (as they can easily do) the method of this book, and apply it to the excellent examples which the Author has supplied, and to those questions set from time to time by the Central Committee, there will be fewer failures at the next examination."—*Extract from Letter of Dr. J. A. McLellan, Senior High School Inspector for Canada, to the "Toronto Mail," November* 22, 1876.

ELEMENTS OF GEOMETRY. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 3*s.* 6*d.*

Containing Books 1 to 6, and portions of Books 11 and 12, of EUCLID, with Exercises and Notes, arranged with the Abbreviations admitted in the Cambridge University and Local Examinations.

Books 1 and 2, limp cloth, 1*s.* 6*d.*, may be had separately.

"Euclid's Axioms will be required, and no proof of any proposition will be admitted which assumes the proof of anything not proved in preceding propositions in Euclid."—*Extract from the Regulations for the Cambridge Local and Schools Examinations for* 1877.

The effect of the above regulation is that the method of proof given in Mr. Hamblin Smith's Geometry satisfies the requirements of these Examinations.

RIVINGTONS' MATHEMATICAL SERIES—continued.

TRIGONOMETRY. *By* J. HAMBLIN SMITH, M.A.
Small 8vo. 4*s*. 6*d*. A KEY. Crown 8vo. 7*s*. 6*d*.

ELEMENTARY STATICS. *By* the same. Small 8vo. 3*s*.

ELEMENTARY HYDROSTATICS. *By* the same. Small 8vo. 3*s*.

A KEY TO STATICS AND HYDROSTATICS. Crown 8vo. 6*s*.

BOOK OF ENUNCIATIONS FOR HAMBLIN SMITH'S GEOMETRY, ALGEBRA, TRIGONOMETRY, STATICS, AND HYDROSTATICS. Small 8vo. 1*s*.

AN INTRODUCTION TO THE STUDY OF HEAT. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*
Small 8vo. 3*s*.

CONTENTS.

General Effects of Heat—Thermometry—Expansion of Gases—Expansion of Solids—Expansion of Liquids—Calorimetry—Latent Heat—Measure of Heat—Diffusion of Heat: Radiation—Convection—Conduction—Formation of Vapour, Dew, &c.; Trade Winds, Ebullition, Papin's Digester, Spheroidal Condition—Congelation—Measurement of Work—Mechanical Equivalent of Heat—Miscellaneous Exercises—Appendix—Index.

[This Edition is adapted for the requirements of the following examinations:—Cambridge University, Cooper's Hill Entrance, Army, Woolwich, Science and Art Department, Oxford and Cambridge Local, Oxford and Cambridge Schools, Civil Service, &c.]

The Principles of Dynamics. An Elementary Text-book for Science Students. *By* R. WORMELL, D.Sc., M.A., *Head-Master of the City of London Middle-Class School.*
Crown 8vo. 6*s*.

Arithmetic, Theoretical and Practical.
By W. H. GIRDLESTONE, M.A., *of Christ's College, Cambridge, Head-Master of Sunningdale Preparatory School, and formerly Principal of the Theological College, Gloucester.*
New Edition, Revised. Crown 8vo. 6*s*. 6*d*.
Also a School Edition. Small 8vo. 3*s*. 6*d*.

SCIENCE

An Easy Introduction to Chemistry.

Edited by the Rev. ARTHUR RIGG, M.A., *and* WALTER T. GOOLDEN, B.A., *Lecturer in Natural Science at Tonbridge School.*
New Edition, revised. With Illustrations. Crown 8vo. 2*s*. 6*d*.

A Year's Botany. Adapted to Home and School Use.

By FRANCES ANNA KITCHENER.
Illustrated by the Author. Second Edition. Crown 8vo. 5*s*.

Notes on Building Construction.

Arranged to meet the requirements of the syllabus of the Science and Art Department of the Committee of Council on Education, South Kensington Museum. Medium 8vo.

PART I.—FIRST STAGE, OR ELEMENTARY COURSE. With 325 woodcuts, 10*s*. 6*d*.

PART II.—COMMENCEMENT OF SECOND STAGE, OR ADVANCED COURSE. With 277 woodcuts, 10*s*. 6*d*.

PART III.—ADVANCED COURSE. With 188 woodcuts. 21*s*.

These notes are intended to furnish a Student with information amply sufficient to enable him to pass the Honours Examination of the Science and Art Department, as far as a knowledge of Building Materials is concerned. They have, however, been extended somewhat beyond what is actually necessary for the mere student, by the addition of tables and information of a practical nature, which it is hoped may be useful to young Engineers, Architects, and others engaged in the design and erection of structures of different kinds. They deal with the nature, characteristics, qualities, and defects of the Materials used in Building and Engineering Works, and they describe the methods of examining and testing such materials. The information given is limited as far as possible to that required by an Engineer, Architect, or Builder, in order to select and understand the materials with which he has to deal. It was originally intended to include in Part III. the information regarding Stresses in parts of Structures, Foundations, Scaffolding, Travellers, &c., required for the advanced course. The bulk of the volume, however, renders it necessary to reserve these subjects to another Part, which will contain as far as possible all the remaining information that is required for the Science Examinations in Building Construction.—*Extract from Preface to Part III.*

REPORT ON THE EXAMINATION IN BUILDING CONSTRUCTION, HELD BY THE SCIENCE AND ART DEPARTMENT, SOUTH KENSINGTON, IN MAY, 1875.—"The want of a text-book in this subject, arranged in accordance with the published syllabus, and therefore limiting the students and teachers to the prescribed course, has lately been well met by a work published by Messrs. Rivingtons, entitled '*Notes on Building Construction*, arranged to meet the requirements of the Syllabus of the Science and Art Department of the Committee of Council on Education, South Kensington.'

(Signed) H. C. SEDDON, Major, R.E."

June 18, 1875. [*Instructor in Construction and Estimating at the School of Military Engineering, Chatham.*]

LATIN

First Latin Writer.

Comprising Accidence, the Easier Rules of Syntax illustrated by copious Examples, and progressive Exercises in Elementary Latin Prose, with Vocabularies.

By GEORGE L. BENNETT, M.A., *Head-Master of the High School, Plymouth; formerly Assistant-Master at Rugby School, and Scholar of St. John's College, Cambridge.*

Crown 8vo. 3*s*. 6*d*.

A KEY for the use of Tutors only. Crown 8vo. 5*s*.

Contents.

PREFACE—ACCIDENCE—EXERCISES ON THE SYNTAX (270): The Simple Sentence: The Compound Sentence: Adjectival Clauses, Adverbial Clauses, Substantival Clauses—LATIN-ENGLISH VOCABULARY—ENGLISH-LATIN VOCABULARY.

"I have prepared the 'First Latin Writer' in the hope that it may prove helpful to those who agree with me that it is quite useless to attempt the difficulties of the Compound Sentence before the Simple Sentence has been thoroughly mastered. The Accidence and Syntax rules are on the lines of the 'Public School Latin Primer'; I have attempted to make them easier for beginners, but little explanation has been given, as the rules are put shortly in plain English. I have not been able to make the disconnected sentences illustrating the Syntax Rules interesting, but I hope the large collection of pieces for translation into Latin will prove so. Difficulties of rare occurrence have been avoided as much as possible"—*Extract from the Preface.*

"The book is a perfect model of what a Latin Writer should be, and is so graduated that from the beginning of a boy's classical course it will serve him throughout till the end as a text-book for Latin prose composition. The exercises, too, are so interesting in themselves, and take up the different idiomatic peculiarities in such an easy and natural way that the pupil almost insensibly comes to be master of them, without having them glaringly thrust upon him in little detached sentences, which, when mixed up in a narrative, he fails, of course, to recognise. We cannot speak too strongly of this little work, and we say to every classical teacher, if you introduce this work into your junior class, you will require no other work throughout till you come to the fifth or sixth form; and perhaps not even then. The book has our unqualified approbation. We ought to mention, for the sake of those who may think of using the work, that there are two sets of vocabularies, which obviate the necessity of having recourse to any Latin dictionary."—*Schoolmaster.*

First Latin Exercises.

Being the Exercises with Syntax Rules and Vocabularies from a "First Latin Writer."

By GEORGE L. BENNETT, M.A., *Head-Master of the High School, Plymouth.* Crown 8vo. 2*s*. 6*d*.

A Second Latin Writer.

By G. L. BENNETT, M.A., *Head-Master of the High School, Plymouth.*

This work, in continuation of the First Latin Writer, will give hints on writing Latin Prose for Boys about to commence the rendering of continuous passages from English Authors into Latin. There will be a large collection of Exercises, graduated according to their difficulty, with Notes.
[*In preparation.*

Easy Latin Stories for Beginners.

By G. L. BENNETT, M.A., *Head-Master of the High School, Plymouth.*

With Vocabularies and Notes. Forming a First Latin Reading Book for Junior Forms in Schools.

New Edition, Revised. Crown 8vo. 2*s.* 6*d.*

A KEY for the use of Tutors only. Crown 8vo. 5*s.*

"These stories are various and amusing, and the grammatical, geographical, and historical notes on them are, as far as we have tested them, careful and judicious."
Saturday Review.

"A more attractive book for very young Latinists we do not remember to have seen."—*Scotsman.*

"We can most cordially recommend Mr. Bennett's little book to all who are engaged in imparting a knowledge of the elements of Latin."—*Liverpool Weekly Albion.*

Caesar. De Bello Gallico. Books I. to III. Also Book I. separately.

Edited, with Preface, Introductions, Maps, Plans, Grammatical, Historical, and Geographical Notes, Indices, Grammatical Appendices, &c., by J. H. MERRYWEATHER, M.A., *and* C. C. TANCOCK, M.A., *Assistant Masters at Charterhouse School.* [*Just ready.*

Small 8vo.

Easy Exercises in Latin Prose.

By CHARLES BIGG, D.D., *Principal of Brighton College.*

Small 8vo. 1*s.* 4*d.*

Latin Prose Exercises.

For Beginners, and Junior Forms of Schools.

By R. PROWDE SMITH, B.A., *Assist.-Master at Cheltenham College.*

New Edition. Crown 8vo. 2*s.* 6*d.*

Henry's First Latin Book.

By THOMAS KERCHEVER ARNOLD, M.A.

Twenty-fourth Edition. 12mo. 3*s.* TUTOR'S KEY. 1*s.*

Arnold's Henry's First Latin Book.

A New and Revised Edition. 12mo. 3*s.*

By C. G. GEPP, M.A., *Head Master of King Edward VI. School, Stratford-upon-Avon; Author of "Progressive Exercises in Latin Elegiac Verse."*

A KEY for the use of Tutors only. Crown 8vo. 5*s.*

(*See Specimen Page opposite.*)

"Whatever may be thought of this work as compared with previous editions, the editor ventures to claim for it the merit of greater simplicity of arrangement. The foot-notes, which have been reserved mainly for explanations of technical words, are fewer, and less crowded with matter; the familiar finger-posts exist no longer; the subtle distinction between numerals above the line with a curve and those without a curve has been abandoned; while a glance at the Table of Contents will show that a definite plan has been followed in the order of the Exercises, the object being to keep in view the stem-formation of words.

"The principal novelties in the present edition are the following:—References have been made throughout to the 'Public School Latin Primer,' the phraseology of which is generally adopted; the formation of words from the Stem is taught from the first; the constructions of the Cases and of the Verb-Noun are fully treated; the Exercises have been almost entirely re-written, and are longer; anecdotes, fables, &c., have been inserted into the passages for translation into English; some fifty test Exercises have been added; and lastly, a General Index has been carefully compiled, by which the search for particular rules or constructions will be greatly facilitated."—*Extract from the Preface.*

CONTENTS.

Cases in Latin and English—On forming the Accusative Case—On the Gender of Substantives—The Genitive and Dative Singular—The Ablative Singular—The Plural Number—On forming Three Tenses in First Conjugation—Accusative with Transitive Verb—On forming Three Tenses in Second Conjugation—On forming Three Tenses in Third Conjugation—On forming Three Tenses in Fourth Conjugation—Supplementary to the foregoing Exercises—The Passive Voice (A-verbs and E-verbs)—The Passive Voice continued (Consonant and U-verbs, I-verbs)—On forming the Perfect, Pluperfect, and Future-Perfect Indicative Active. (Adjectives in *-x*, *-ns*, *-rs*)—The Comparative and Superlative of Adjectives. (*Acer, Unus*)—The Infinitive—The Infinitive continued. *Sum, Possum, Volo.* (The Prolative Infinitive)—The Infinitive continued. Gerunds. (*Personal and Demonstrative Pronouns*)—The Gerundive. Gerundive Attraction. (*Dative after Agent*)—The Supines. Eo. (*Nē—quidem*)—The Participles. Periphrastic Conjugation—The Perfect and Pluperfect Passive. Deponents—The Accusative and Infinitive—The Relative Pronoun. Copulative Verbs—The Reflexive Pronouns. *Ipse. Idem*—The Subjunctive. Ut and Nĕ expressing Purpose. (*Consecution of Tenses*)—The Subjunctive continued. Ut expressing a Consequence.' (*Quis, quæ, quid*)—The Imperative. *Ne* in Prohibitions—The Indefinite Pronoun Quis—Cum (Quum). (*Interrogative Sentences*)—Impersonal Verbs—Apposition. *Summus, Imus*, etc. (Conditional Sentences)—The Accusative. Place. Space. Time. Double Accusative. Prepositions—The Genitive (*Subjective Genitive*)—The Genitive continued (*Objective Genitive*). Miseret, pænitet, etc.—The Dative—The Dative continued. Impersonal Construction—The Ablative. Quasi-Passive Verbs—The Ablative continued—*Time, Price*—The Ablative continued. *Place.* (The Locative)—The Ablative Absolute—Ablative of the thing compared. *Quam* in Comparison—Ablative denoting Amount of Difference. Tanto—quanto; eo—quo)—Translation of '*May*,' '*Might*,' '*Ought*'—Ut, Nĕ, with Verbs of Fearing—Quin. Quominus—Interrogatives. Direct Questions. Indirect Questions. Double Questions—Some Conjunctions always found with the Subjunctive—Qui with the Subjunctive—Transitive and Intransitive Verbs—Various uses of Qui—*Either—or. Whether—or*—Participial Constructions—Conjunctions.

Memorabilia—Supplementary Exercises (65-112)—Differences of Idiom—Table of Synonymes—Questions on Syntax—Appendix—Vocabulary I., English-Latin—Vocabulary II., Latin-English—General Index.

LESSON 5.

The Ablative Singular.

35. The following are the *Ablative endings* of the five declensions in the Singular Number.

I.	II.	III.	IV.	V.
-ā	-ō	-ĕ(-i)	-ū	-ē

Nouns which make Acc. in *-im*, and neuter nouns ending in *e*, *al*, *ar*, have Ablative in -i. For list of nouns and exceptions, see Appendix, IV.

The Ablative is formed in each declension by adding the *endings* according to the rule given in **15**, **16**, and **29**.

36. The *most common* signs of the Latin Ablative case are *by*, *with*, *from*. There are, however, many other signs, *e.g.*, *at*, *in*, *for*, *of*, *than*, *upon*, which will be noticed hereafter.

37. The following Prepositions, governing the Ablative, are used with Substantives which signify *persons* or *living beings;* viz., *a* (or *ab* before a vowel), meaning *by;* *cum*, meaning *with* (= together with); and *a* (*ab*), *e* (or *ex* before a vowel), meaning *from* (Appendix, XXIII, β).

'*By*' is the proper sign of the Ablative of the *Agent* * or '*doer*,' whether a *person* or a *living being.* The Agent is *never* put in the Ablative *without* the Prep. *a* or *ab.*

'*With*' is the proper sign of the Ablative of the *Instrument*, or '*thing by means of which*' anything is done. The Ablative of the Instrument is put *without* a Preposition.

EXERCISE 5.

38. *Vocabulary* 5.

		Gen. Pl.
brother,	frater,	fratRum.
exile,	exsul,	exsŭLum.
labour,	labor,	labōRum.

[GEPP'S ARNOLD'S FIRST LATIN BOOK. *See* p. 20.]

A Practical Introduction to Latin Prose Composition.

By THOMAS KERCHEVER ARNOLD, M.A.
Eighteenth Edition. 8vo. 6*s*. 6*d*. TUTOR'S KEY. 1*s*. 6*d*.

Arnold's Practical Introduction to Latin Prose Composition.

A New and Revised Edition. 8vo.
By GEORGE G. BRADLEY, M.A., *Master of University College, Oxford, and late Head-Master of Marlborough College.*
A KEY for the use of Tutors only. [*In preparation.*

Exercises on the Elementary Principles of Latin Prose Composition.

With Examination Papers on the Elementary Facts of Latin Accidence and Syntax.
By J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*
Crown 8vo. 3*s*. 6*d*.
A KEY for the use of Tutors only. Crown 8vo. 5*s*.

Progressive Exercises in Latin Elegiac Verse.

By C. G. GEPP, M.A., *Head Master of King Edward VI. School, Stratford-upon-Avon.*
Fourth Edition, Revised. Crown 8vo. 3*s*. 6*d*. TUTOR'S KEY, 5*s*.

A First Verse Book.

Being an Easy Introduction to the Mechanism of the Latin Hexameter and Pentameter.
By THOMAS KERCHEVER ARNOLD, M.A.
Eleventh Edition. 12mo. 2*s*. TUTOR'S KEY, 1*s*.

The Æneid of Vergil.

Edited, with Notes at the end, by FRANCIS STORR, M.A., *Chief Master of Modern Subjects at Merchant Taylors' School.*

Crown 8vo.

BOOKS I. and II. *2s. 6d.* BOOKS III. and IV. [*In preparation.*
BOOKS XI. and XII. *2s. 6d.*

"Altogether, his taste and judgment thus far give earnest of a strong editor of the Æneid."—*Saturday Review.*

"This is a very complete, scholarly, and convenient edition of the Æneid, and we have pleasure in recommending it to the notice of teachers."—*Educational Times.*

"To produce a good school edition of Virgil is now not difficult, but Mr. Storr has performed the task with more than ordinary skill. . . . . . But the great merit of this little book is in the number of apposite quotations from modern authors, which are adduced to illustrate Virgil's thoughts or style of expression. . . . . . There is a piquancy about such notes as this which cannot fail to be attractive to schoolboys."—*Athenæum.*

Stories from Ovid in Elegiac Verse.

With Notes for School Use and Marginal References to the PUBLIC SCHOOL LATIN PRIMER.

By R. W. TAYLOR, M.A., *Head-Master of Kelly College, Tavistock, and late Fellow of St. John's College, Cambridge.*

New and Revised Edition. Crown 8vo. *3s. 6d.*

Eclogæ Ovidianæ.

From the Elegiac Poems. With English Notes.

By THOMAS KERCHEVER ARNOLD, M.A.

Fourteenth Edition, Revised. 12mo. *2s. 6d.*

Cicero de Amicitiâ.

Edited, with Notes and an Introduction, by ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.*

Small 8vo. *2s.*

CONTENTS.

"No volume on our list is more valuable than Mr. Sidgwick's edition of Cicero's treatise 'De Amicitia,' prefaced by a review of the circumstances and scheme and interlocutors of the dialogue, a conspectus and analysis of the same, and an excellent appendix on the scheme of the subjunctive, which cannot fail to be useful to schoolboys and students. The 'De Amicitia' is just the work to be placed in a young student's hands for translation and retranslation; and Mr. Sidgwick's explanatory and illustrative notes are calculated to fix its matter in the memory. . . . . We can strongly recommend this in every respect well-furnished edition."—*Saturday Review.*

Cicero de Senectute, with Introduction and Notes.

Forming a Volume of Arnold's School Classics.

12mo. 2*s*. 6*d*.

Classical Examination Papers.

Edited, with Notes and References, by P. J. F. GANTILLON, M.A., *Classical Master at Cheltenham College.*

Second Edition. Crown 8vo. 7*s*. 6*d*.

Or interleaved with writing-paper, half-bound, 10*s*. 6*d*.

Elementary Rules of Latin Pronunciation.

By ARTHUR HOLMES, M.A., *late Senior Fellow and Dean of Clare College, Cambridge.*

Crown 8vo. On a card, 9*d*.

Outlines of Latin Sentence Construction.

By E. D. MANSFIELD, M.A., *Assistant-Master at Clifton College.*

Demy 8vo. On a card, 1*s*.

Materials and Models for Latin Prose Composition.

Selected and arranged by J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford; and* T. F. DALLIN, M.A., *Tutor, late Fellow, of Queen's College, Oxford.*

New Edition, re-arranged, with fresh Pieces and additional References.

Crown 8vo. 6*s*. 6*d*.

Latin Version of (60) Selected Pieces from Materials and Models.

By J. Y. SARGENT, M.A.

Crown 8vo. 5*s*.

May be had by Tutors only, on direct application to the Publishers.

Selections from Livy, Books VIII. and IX.

With Notes and Map.

By E. CALVERT, LL.D., *St. John's College, Cambridge; and* R. SAWARD, M.A., *Fellow of St. John's College, Cambridge; Assistant-Master at Shrewsbury School.*

Small 8vo. 2*s.*

Cornelius Nepos.

With Critical Questions and Answers, and an Imitative Exercise on each Chapter.

By THOMAS KERCHEVER ARNOLD, M.A.

Fifth Edition. 12mo. 4*s.*

Terenti Comoediae.

Edited by T. L. PAPILLON, M.A., *Fellow of New College.*

Crown 8vo.

ANDRIA ET EUNUCHUS. With Introduction on Prosody. 4*s.* 6*d.*
Or separately, ANDRIA. With Introduction on Prosody. 3*s.* 6*d.*
EUNUCHUS. 3*s.*

Juvenalis Satirae.

Edited by G. A. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*
THIRTEEN SATIRES.

Second Edition. Crown 8vo. 5*s.*

Persii Satirae.

Edited by A. PRETOR, M.A., *of Trinity College, Cambridge.*

Crown 8vo. 3*s.* 6*d.*

Horati Opera.

By J. M. MARSHALL, M.A., *Under-Master at Dulwich College.*
VOL. I.—THE ODES, CARMEN SECULARE, AND EPODES.

Crown 8vo. 7*s.* 6*d.*

Taciti Historiae.

Edited by W. H. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*

Crown 8vo. 6*s.*

BOOKS I. and II., 6*s.* BOOKS III., IV., and V., 6*s.*

GREEK

A Primer of Greek Accidence for the Use of Schools.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford; and* E. D. MANSFIELD, M.A., *Assistant-Master at Clifton College. With a Preface by* JOHN PERCIVAL, M.A., LL.D., *President of Trinity College, Oxford; late Head-Master of Clifton College.*

Second Edition, Revised. Crown 8vo. *2s. 6d.*

(*See Specimen Page opposite.*)

"A glance at the book will show that the Editors, remembering how important it is that the early training should run on the same lines as the higher studies that are to follow, have kept steadily in view its preparatory character. They have at the same time bestowed much pains on making it as clear and intelligible as possible, whilst they have given special prominence to the laws that regulate the changes of sound. The learner's attention is also specially drawn to the Stem-theory, particularly in dealing with the various parts of the Verb and their relation to each other, and in the classification of the Irregular Verbs."—*Extract from the Preface.*

A Primer of Greek Syntax.

By the same Editors. [*In preparation.*

A Practical Greek Method for Beginners.

Being a graduated application of Grammar to Translation and Composition.

By F. RITCHIE, M.A., *and* E. H. MOORE, M.A., *Assistant-Masters at the High School, Plymouth.* [*Nearly ready.*

The aim of this book, which is at once a Grammar and Exercise Book, is to secure an uniform method of teaching Grammar, and to afford abundant practice in inflexion, &c., at the time that the Grammar is being learnt.

PART I. contains the Substantives, Adjectives, Pronouns, and Regular Pure Verbs with Exercises (English-Greek and Greek-English), introducing the main rules of Syntax of the Simple Sentence.

PART II. contains the remaining Classes of Regular and Irregular Verbs, with Consecutive Pieces for Translation, introducing progressively the Syntax of the Compound Sentence.

DECLENSION OF SUBSTANTIVES.

36. SECOND DECLENSION.

O STEMS.

	Simple.			Contracted.			
Stem. Engl.	λογο, *speech.*	νησο, *island.*	ζυγο, *yoke.*	νοο, *mind.*		οστεο, *bone.*	
Sing. *Nom.*	ὁ λόγος	ἡ νῆσος	τὸ ζυγόν	ὁ νόος	νοῦς	τὸ ὀστέον	ὀστοῦν
Voc.	λόγε	νῆσε	ζυγόν	νόε	νοῦ	ὀστέον	ὀστοῦν
Acc.	λόγον	νῆσον	ζυγόν	νόον	νοῦν	ὀστέον	ὀστοῦν
Gen.	λόγου	νήσου	ζυγοῦ	νόου	νοῦ	ὀστέου	ὀστοῦ
Dat.	λόγῳ	νήσῳ	ζυγῷ	νόῳ	νῷ	ὀστέῳ	ὀστῷ
Dual *N. V. A.*	λόγω	νήσω	ζυγώ	νόω	νώ	ὀστέω	ὀστώ
G. D.	λόγοιν	νήσοιν	ζυγοῖν	νόοιν	νοῖν	ὀστέοιν	ὀστοῖν
Plur. *N. V.*	λόγοι	νῆσοι	ζυγά	νόοι	νοῖ	ὀστέα	ὀστᾶ
Acc.	λόγους	νήσους	ζυγά	νόους	νοῦς	ὀστέα	ὀστᾶ
Gen.	λόγων	νήσων	ζυγῶν	νόων	νῶν	ὀστέων	ὀστῶν
Dat.	λόγοις	νήσοις	ζυγοῖς	νόοις	νοῖς	ὀστέοις	ὀστοῖς

EXAMPLES.

SIMPLE.—ἄνθρωπος, ὁ, *man;* οἶκος, ὁ, *house;* ξύλον, τό, *wood.*
CONTR. —πλοῦς, ὁ, *voyage;* κανοῦν, τό, *basket.*

Obs. 1. In the neuters, nom., acc., and voc. are always the same; and in the plural these cases always end in *α*. The contraction of ὀστέα into ὀστᾶ is irregular, cp. 11.

Obs. 2. The following words are feminine :—ὁδός, *way;* νῆσος, *island;* νόσος, *disease;* δρόσος, *dew;* σποδός, *ashes;* ψῆφος, *pebble;* ἄμπελος, *vine;* γνάθος, *jaw;* ἤπειρος, *continent;* and some others.

A First Greek Writer.

By ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.*

[*In preparation.*

An Introduction to Greek Prose Composition, with Exercises.

By ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.*

Second Edition, Revised. Crown 8vo. 5*s*.

A KEY for the use of Tutors only. 5*s*.

(*See Specimen Page opposite.*)

CONTENTS.

(1) Notes on Constructions: Preliminary—Final Sentences—Deliberative Sentences—Conditional Sentences—Indefinite Sentences—Oratio Obliqua—Consecutive Sentences—Limitative Sentences—Temporal Sentences—Casual Sentences—Concessive Sentences. (2) Notes on Idiom: Preliminary—Abstract and Concrete—Tense—Negatives—Connection—Tenses—Attraction—Participles—Middle Voice—Metaphors—175 Exercises. Lists: Pronouns, &c.—Conjunctions—Participles—Prepositions—Scheme of Subjunctive—Optative—Index of Moods. Vocabulary, &c.

SCENES FROM GREEK PLAYS

RUGBY EDITION

Abridged and adapted for the use of Schools, by

ARTHUR SIDGWICK, M.A.,

ASSISTANT-MASTER AT RUGBY SCHOOL, AND LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

Small 8vo. 1*s*. 6*d*. each.

ARISTOPHANES.

THE CLOUDS. THE FROGS. THE KNIGHTS. PLUTUS.

EURIPIDES.

IPHIGENIA IN TAURIS. THE CYCLOPS. ION.
ELECTRA. ALCESTIS. BACCHÆ. HECUBA.

---

trace of anything artificial, except perhaps in the orators: and even there the art is shown as much in the *extreme naturalness* of the order as in anything else.

The considerations therefore that determine the order of words are chiefly the following: clearness: emphasis: neatness and euphony.

Clearness is the chief thing. Let the words come out in their natural order, but so that there be no ambiguity. In a Latin sentence you have to think about balance and point and marshalling of verbs and so forth: in Greek it is best to be not hampered by rules for order, but to strive simply to say what you mean, and let it come out in the most natural way; and above all, to be clear.

An extremely good test for Greek prose composition is to leave it for a bit after writing, and then read it all over like a new piece. If you are stopped for an instant by not seeing the meaning, or are for an instant misled, then be sure there is a blemish in the order or clearness of the writing.

A common mistake for beginners to make in Greek is to be *artificial* in the arrangement of sentences: to start with some theory, as for example that notion (derived from Latin) that all verbs must be at the ends of the clauses. And so if they get a sentence to translate like this:—

'He said he would kill all who did not do what he ordered,'

They will produce the following obscure passage:

οὗτος, ὅτι πάντας, οἳ μὴ ὅπερ κελεύοι δρῷεν, ἀποκτενοῖ, ἔφη, which is perfectly correct in Grammar, but the order is dreadful, with that heavy *sediment* of verbs at the end.

[INTRODUCTION TO GREEK PROSE—SIDGWICK. See p. 28.]

Stories in Attic Greek.

Forming a Greek Reading Book for the use of Junior Forms in Schools. With Notes and Vocabulary.

By FRANCIS DAVID MORICE, M.A., *Assistant-Master at Rugby School; and Fellow of Queen's College, Oxford.*

Crown 8vo. 3*s*. 6*d*.

(*See Specimen Page opposite.*)

Contents.

Hints to Beginners—How to look out words in a Vocabulary—Stems—Augments—Temporal Augments—Compound Verbs—Changes of Prepositions in Compound Verbs—Special Irregularities—List of Changes of Prepositions in Composition—Hints on Construing—Structure of Sentences—Conjunctions, &c.—Stops—Pronouns—Articles: (1) Marking subject; (2) Words placed between Article and Noun; (3) Repetition of Article; (4) Article with a Participle; (5) Article equivalent to a Possessive Pronoun; (6) Article with Infinitive—250 Stories—Notes—Index to Stories—Vocabulary of Proper Names—General Vocabulary.

"The plan of the book is admirable. . . . . The learner is assisted by a number of ingenious devices. The augments and prepositions in the text are divided from the stem by hyphens, so as to make the words easy to look out. The text is preceded by admirably prepared Hints to Beginners, which tell the learner in easy language all he need know about stems, augments, compound verbs, proper names, &c. The Hints on Construing are excellent, and are the best remedy against the use of a translation. The Vocabulary at the end of the book is full of instruction, and the notes are sufficient and clear. Such a book, thoroughly mastered, should lead the learner by easy steps from his grammar to his Thucydides."—*Academy.*

"The idea is a good one, and it has been admirably carried out. Of the stories, some have been judiciously selected from the best-read Attic writers, and carefully adapted to the end in view, and others 'appear now for the first time in a Greek dress.' The whole have been arranged on the progressive principle, so that the learner is gradually led on from simple and easy to more complicated and difficult constructions. Such, indeed, is the interest of the stories that it would be more correct to say that he is lured on. The text is beautifully printed, and has the characteristic feature of separating by a hyphen the preposition and the augment from the verb to which they are prefixed—an arrangement which will greatly facilitate the use of the vocabulary. The vocabularies and notes are all that could be desired; and the practical 'Hints to Beginners' on the right methods of using the vocabulary and of construing, if attended to, will save the learner much useless and troublesome research."

Scotsman.

The Anabasis of Xenophon.

Edited, with Preface, Introduction, Historical Sketch, Itinerary, Syntax Rules, Notes, Indices, Vocabularies, and Maps, by R. W. TAYLOR, M.A., *Head-Master of Kelly College, Tavistock, and late Fellow of St. John's College, Cambridge.*

Crown 8vo.

Books I. and II. 3*s*. 6*d*. Books III. and IV. 3*s*. 6*d*.

A Short Greek Syntax.

Extracted from "XENOPHON'S ANABASIS, WITH NOTES."

By R. W. TAYLOR, M.A., *Head-Master of Kelly College, Tavistock.*

Crown 8vo. 9*d*.

THE DOLPHINS.

82. Κοίρανος ὁ Μιλήσιος ἁλιέας τινὰς εἶδέ ποτε, δελφῖνα τῷ δικτύῳ λαβόντας, καὶ μέλλοντάς αὐτὸν κατα-κόψειν. προς-ελθὼν οὖν τούτοις ἀργύριον ἐ-δίδου, καὶ πριάμενος τὸν δελφῖνα ἀφ-ῆκεν αὐτὸν πάλιν εἰς τὴν θάλασσαν. μετὰ δὲ ταῦτα πλοῦν ποιούμενος ὁ Κοίρανος ναυαγίᾳ ἐ-χρήσατο. καὶ οἱ μὲν ἄλλοι πάντες ἀπ-ε-πνίγησαν· τὸν δὲ Κοίρανον δελφὶς ἀπροσδοκήτως παρα-γενόμενος ἔ-σωσε. γηράσας δὲ ὕστερον ὁ ἄνθρωπος ἐν τῇ ἑαυτοῦ πατρίδι ἐ-τελεύτησεν. ἡ δὲ ἐκφορὰ αὐτοῦ ἐν αὐτῷ τῷ αἰγιαλῷ κατὰ τύχην ἐ-γένετο. ἐ-φάνη οὖν ἐν ἐκείνῃ τῇ ἡμέρᾳ πλῆθος πολὺ δελφίνων ἐν τῷ λιμένι, συν-ακολουθούντων τοῖς τὸν νεκρὸν ἐκ-φέρουσιν, ὡσεὶ συν-εκ-φερόντων αὐτὸν, καὶ συγ-κλαιόντων.

THE BELL-TOWER.

83. ἦν ποτε ἐν πόλει τινὶ κώδων ἐν πύργῳ ἀνα-κρεμασθείς. νόμος δὲ ἐ-γένετο ἐκ παλαιοῦ τοῖς πολίταις τοὺς ἄδικόν τι παθόντας προς-ιέναι ἐκεῖσε, καὶ τὸ σπάρτον ἐπι-σπᾶν, ὥστε τὸν κώδωνα κινούμενον ψοφεῖν. ἔπειτα δὲ ἔ-δει τοὺς ἄρχοντας συν-έρχεσθαι εὐθὺς εἰς τὸ δικαστήριον, καὶ τὸ πρᾶγμα ἀκούσαντας δικάζειν.

ἄνθρωπος δέ τις τῶν πολιτῶν ἵππον ἔχων, τοῦτον οὐκ ἤθελε γηράσαντα τρέφειν, ἀλλὰ λύσας αὐτὸν ἀπὸ τῆς φάτνης ἀπ-ήλασεν. πλανώμενος οὖν ὁ ἵππος περὶ πᾶσαν τὴν πόλιν τέλος εἰς τὸν πύργον ἐφ-ίκετο. προς-ελθὼν δὲ ἐνταῦθα πρὸς τὴν θύραν, καὶ τὸ σπάρτον ἰδὼν, ἤρξατο αὐτὸ τρώγειν διὰ λιμὸν, ὁ δὲ κώδων εὐθὺς ἐ-ψόφησεν.

84. συν-αθροισθέντες τοίνυν εἰς τὸ δικαστήριον οἱ ἄρχοντες ἐ-κέλευον τὸν ἀδικηθέντα παρ-ιέναι κατὰ τὸν

Elements of Greek Accidence.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford, and late Assistant-Master at Clifton College.*

Crown 8vo. 4*s.* 6*d.*

Selections from Lucian.

With English Notes.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford, and late Assistant-Master at Clifton College.*

Second Edition. Small 8vo. 3*s.* 6*d.*

Zeugma; or, Greek Steps from Primer to Author.

By the Rev. LANCELOT SANDERSON, M.A., *Principal of Elstree School, late Scholar of Clare College, Cambridge; and the* Rev. F. B. FIRMAN, M.A., *Assistant-Master at Elstree School, late Scholar of Jesus College, Cambridge.*

Second Edition, Revised. Small 8vo. 1*s.* 6*d.*

A Practical Introduction to Greek Prose Composition.

By THOMAS KERCHEVER ARNOLD, M.A.

Twelfth Edition. 8vo. 5*s.* 6*d.* TUTOR'S KEY, 1*s.* 6*d.*

Arnold's Practical Introduction to Greek Prose Composition.

By EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

A New and Revised Edition. 8vo. [*In the Press.*

The First Greek Book.

On the plan of *Henry's First Latin Book.*

By THOMAS KERCHEVER ARNOLD, M.A.

Sixth Edition. 12mo. 5*s.* TUTOR'S KEY, 1*s.* 6*d.*

A Practical Introduction to Greek Accidence.

By THOMAS KERCHEVER ARNOLD, M.A.

Ninth Edition. 8vo. 5*s.* 6*d.*

The Antiquities of Greece. Translated from the German of G. F. SCHOEMANN.

By E. G. HARDY, M.A., *late Fellow of Jesus College, Oxford, and* J. S. MANN, M.A., *sometime Scholar of Exeter College, Oxford.*

2 Vols. 8vo. [*Vol. I. nearly ready.*

VOLUME I. contains (1) Homeric Greece, (2) Historical Greece. Historical Greece is divided into the three following divisions :—

PART I.—GENERAL CHARACTERISTICS OF THE GREEK STATE :—Difference of race.—Idea and conditions of the State—Chief forms of constitution—The citizens and the labouring classes — System of public discipline, &c., &c. PART II.—HISTORICAL STATEMENTS WITH RESPECT TO THE CONSTITUTIONS OF PARTICULAR STATES : Monarchy—Oligarchy—Lawgivers — Tyrants — Democracy — Theoretical Reformers — Party struggles, &c., &c. PART III.—DESCRIPTION OF PRINCIPAL STATES IN DETAIL : Political, social, military, and financial condition of Sparta, Crete, and Athens. SPARTA : Spartiatæ—Helots—Legislation of Lycurgus—Military system—Hellenic policy, &c. ATHENS : Land and People—Solonian legislation—Development of Democracy—Slaves—Financial system—Judicial system, &c.

VOLUME II. will contain : PART I.—INTERNATIONAL RELATIONS. PART II.—GREEK RELIGION.

Alexander the Great in the Punjaub.

Adapted from Arrian, Book V. An easy Greek Reading Book.

Edited, with Notes and a Map, by the Rev. CHARLES E. MOBERLY, M.A., *Assistant-Master at Rugby School, and formerly Scholar of Balliol College, Oxford.*

Small 8vo. 2*s.*

Stories from Herodotus.

The Tales of Rhampsinitus and Polycrates, and the Battle of Marathon and the Alcmæonidae. *In Attic Greek.*

Edited by J. SURTEES PHILLPOTTS, M.A., *Head-Master of Bedford Grammar School; formerly Fellow of New College, Oxford.*

Crown 8vo. 1*s.* 6*d.*

Iophon: an Introduction to the Art of Writing Greek Iambic Verses.

By the WRITER *of "Nuces" and "Lucretilis."*

Crown 8vo. 2*s.*

Homer's Iliad.

Edited, with Notes at the end for the Use of Junior Students, by ARTHUR SIDGWICK, M.A., *Assistant-Master at Rugby School, and late Fellow of Trinity College, Cambridge.*

Small 8vo.

Books I. and II. 2*s.* 6*d.*
Books III. and IV. [*In preparation.*

CONTENTS:—Preface—Introduction—The Language of Homer—The Dialect—Forms—Syntax—General Text, Books I. and II.—Notes—Indices.

"Mr. Arthur Sidgwick's edition of Books I. and II. of the 'Iliad' is a volume, for its size, well equipped with every kind of legitimate help to young Homeric students; we must especially mention the pains he has taken in clearing away the difficulties of the strange and epic forms of words by bringing them together and enclosing them in brackets, his almost excessive liberality in the matter of 'Indices,' and his practical supersession of a separate lexicon or dictionary."—*Saturday Review.*

"Mr. Sidgwick has converted the first two books of Homer's 'Iliad' into an excellent school book. The text, printed in clear Greek type, is broken up into small sections, each with an appropriate heading. Prefixed is a detailed account of the distinctive forms, both verbal and syntactic, of the Homeric dialect. Appended are copious explanatory and illustrative notes, which ought to make the learner's path as easy as it is at all desirable. In these notes the scholar's attention is kept constantly directed to the Homeric deviations from the standard Attic forms. The geographical and mythological information conveyed in these notes is minute and accurate. The work is also furnished with several Indices, which will enable the learner with the greatest ease to turn up any word or passage to which he may have occasion to refer. A short but interesting introduction gives a summary of what is known of the literary history of the Homeric poems, an outline of the story of the 'Iliad,' and a descriptive catalogue of the Homeric deities."—*Scotsman.*

Homer without a Lexicon, for Beginners.

ILIAD, Book VI.

Edited, with Notes giving the meanings of all the less common words, by J. SURTEES PHILLPOTTS, M.A., *Head Master of Bedford Grammar School, formerly Fellow of New College, Oxford.*

Small 8vo, 2*s.*

Homer for Beginners.

ILIAD, Books I.—III. With English Notes.
By THOMAS KERCHEVER ARNOLD, M.A.
Fifth Edition. 12mo. 3*s.* 6*d.*

The Iliad of Homer.

With English Notes and Grammatical References.
By THOMAS KERCHEVER ARNOLD, M.A.
Fifth Edition. 12mo. 12*s.*

The Iliad of Homer.

From the Text of Dindorf. With Preface and Notes.

By S. H. REYNOLDS, M.A., *Fellow and Tutor of Brasenose College, Oxford.*

Books I.—XII. Crown 8vo. 6*s.*

A Complete Greek and English Lexicon for the Poems of Homer and the Homeridæ.

By G. CH. CRUSIUS. *Translated from the German. Edited by* T. K. ARNOLD, M.A.

New Edition. 12mo. 9*s.*

An Elementary Greek Grammar.

By J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*

Crown 8vo. 4*s.* 6*d.*

An Easy Greek Reading Book,

By C. HEPWORTH GIBSON, M.A., *Assistant-Master at Merchant Taylors' School.* [*In preparation.*

Xenophon's Memorabilia.

Book I., with a few omissions. *Edited, with an Introduction and Notes, by the* Rev. C. E. MOBERLY, *Assistant-Master at Rugby School, and formerly Scholar of Balliol College, Oxford.*

Small 8vo. 2*s.*

Classical Examination Papers.

Edited, with Notes and References, by P. J. F. GANTILLON, M.A., *Classical Master at Cheltenham College.*

Second Edition. Crown 8vo. 7*s.* 6*d.*

Or interleaved with writing-paper, half-bound, 10*s.* 6*d.*

Materials and Models for Greek Prose Composition.

Selected and arranged by J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford; and* T. F. DALLIN, M.A., *Tutor, late Fellow, of Queen's College, Oxford.*

Second Edition, containing Fresh Pieces and additional References.

Crown 8vo. 5*s.*

Greek Version of Selected Pieces from Materials and Models.

By J. Y. SARGENT, M.A.

Crown 8vo. 7*s.* 6*d.*

May be had by Tutors only, on direct application to the Publishers.

Herodoti Historia.

Edited by H. G. WOODS, M.A., *Fellow and Tutor of Trinity College, Oxford.*

Crown 8vo.

BOOK I. 6*s.* BOOK II. 5*s.*

Isocratis Orationes.

Edited by JOHN EDWIN SANDYS, M.A., *Fellow and Tutor of St. John's College, Cambridge, and Public Orator of the University.*

Crown 8vo.

AD DEMONICUM ET PANEGYRICUS. 4*s.* 6*d.*

Demosthenes.

Edited, with English Notes and Grammatical References, by THOMAS KERCHEVER ARNOLD, M.A.

12mo.

OLYNTHIAC ORATIONS. Third Edition. 3*s.*
PHILIPPIC ORATIONS. Third Edition. 4*s.*
ORATION ON THE CROWN. Second Edition. 4*s.* 6*d.*

Demosthenis Orationes Privatae.

Edited by ARTHUR HOLMES, M.A., *late Senior Fellow and Dean of Clare College, Cambridge, and Preacher at the Chapel Royal, Whitehall.*

Crown 8vo.

DE CORONA. 5*s.*

Demosthenis Orationes Publicae.

Edited by G. H. HESLOP, M.A., *late Fellow and Assistant-Tntor of Queen's College, Oxford; Head-Master of St. Bees.*

Crown 8vo.

OLYNTHIACS, 2*s*. 6*d*. } or, in One Volume, 4*s*. 6*d*.
PHILIPPICS, 3*s*. }
DE FALSA LEGATIONE, 6*s*.

Sophocles.

Edited by T. K. ARNOLD, M.A., ARCHDEACON PAUL, *and* HENRY BROWNE, M.A.

12mo.

AJAX. 3*s*. PHILOCTETES. 3*s*. ŒDIPUS TYRANNUS. 4*s*.

Sophoclis Tragoediae.

Edited by R. C. JEBB, M.A., *Professor of Greek at the University of Glasgow, late Fellow and Tutor of Trinity College, Cambridge.*

Crown 8vo.

ELECTRA. 3*s*. 6*d*. AJAX. 3*s*. 6*d*.

Aristophanis Comoediae.

Edited by W. C. GREEN, M.A., *late Fellow of King's College, Cambridge; Assistant-Master at Rugby School.*

Crown 8vo.

THE ACHARNIANS and THE KNIGHTS. 4*s*.
THE CLOUDS. 3*s*. 6*d*. THE WASPS. 3*s*. 6*d*.
THE ACHARNIANS and THE KNIGHTS, revised for Schools. 4*s*.

An Introduction to Aristotle's Ethics.

Books I.—IV. (Book X., c. vi.—ix. in an Appendix). With a Continuous Analysis and Notes. Intended for the use of Beginners and Junior Students.

By the Rev. EDWARD MOORE, B.D., *Principal of S. Edmund Hall, and late Fellow and Tutor of Queen's College, Oxford.*

Second Edition, Revised and Enlarged. Crown 8vo. 10*s*. 6*d*.

Aristotelis Ethica Nicomachea.

Edidit, emendavit, crebrisque locis parallelis e libro ipso, aliisque ejusdem Auctoris scriptis, illustravit JACOBUS E. T. ROGERS, A.M.

Small 8vo. 4*s*. 6*d*. Interleaved with writing-paper, half-bound. 6*s*.

Thucydidis Historia. Books I. and II.

Edited by CHARLES BIGG, D.D., *late Senior Student and Tutor of Christ Church, Oxford; Principal of Brighton College.*

Crown 8vo. 6*s*.

Thucydidis Historia. Books III. and IV.

Edited by G. A. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*

Crown 8vo. 6*s*.

Selections from Aristotle's Organon.

Edited by JOHN R. MAGRATH, D.D., *Provost of Queen's College, Oxford.*

Second Edition. Crown 8vo. 3*s*. 6*d*.

A Copious Phraseological English-Greek Lexicon.

Founded on a work prepared by J. W. FRÄDERSDORFF, Ph.D., *late Professor of Modern Languages, Queen's College, Belfast.*

Revised, Enlarged, and Improved by the late THOMAS KERCHEVER ARNOLD, M.A., *and* HENRY BROWNE, M.A.

Fifth Edition. 8vo. 21*s*.

Madvig's Syntax of the Greek Language, especially of the Attic Dialect.

For the use of Schools.

Edited by THOMAS KERCHEVER ARNOLD, M.A.

Second Edition. Imperial 16mo. 8*s*. 6*d*.

Recommended by the Cambridge Board of Classical Studies for the Classical Tripos.

Notes on the Greek Text of the Acts of the Apostles.

By J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, Cambridge.*

Crown 8vo. 4*s*. 6*d*.

The Greek Testament.

With a Critically Revised Text; a Digest of Various Readings; Marginal References to Verbal and Idiomatic Usage; Prolegomena; and a Critical and Exegetical Commentary. For the use of Theological Students and Ministers.

By HENRY ALFORD, D.D., *late Dean of Canterbury.*

New Edition. 4 vols. 8vo. 102*s.*

The Volumes are sold separately, as follows:—

Vol. I.—The FOUR GOSPELS. 28*s.*
Vol. II.—ACTS to 2 CORINTHIANS. 24*s.*
Vol. III.—GALATIANS to PHILEMON. 18*s.*
Vol. IV.—HEBREWS to REVELATION. 32*s.*

The Greek Testament.

With Notes, Introductions, and Index.

By CHR. WORDSWORTH, D.D., *Bishop of Lincoln.*

New Edition. 2 vols. Impl. 8vo. 60*s.*

The Parts may be had separately, as follows:—

The GOSPELS. 16*s.*
The ACTS. 8*s.*
St. Paul's EPISTLES. 23*s.*
GENERAL EPISTLES, REVELATION, and INDEX. 16*s.*

Notes on the Greek Testament.

By the Rev. ARTHUR CARR, M.A., *Assistant-Master at Wellington College, late Fellow of Oriel College, Oxford.*

THE GOSPEL ACCORDING TO S. LUKE.

Crown 8vo. 6*s.*

CATENA CLASSICORUM

Crown 8vo.

Sophoclis Tragoediae. *By* R. C. JEBB, M.A.
THE ELECTRA. 3*s*. 6*d*. THE AJAX. 3*s*. 6*d*.

Juvenalis Satirae. *By* G. A. SIMCOX, M.A. 5*s*.

Thucydidis Historia.—Books I. & II.
By CHARLES BIGG, D.D. 6*s*.

Thucydidis Historia.—Books III. & IV.
By G. A. SIMCOX, M.A. 6*s*.

Demosthenis Orationes Publicae. *By* G. H. HESLOP, M.A.
THE OLYNTHIACS. 2*s*. 6*d*. } or, in One Volume, 4*s*. 6*d*.
THE PHILIPPICS. 3*s*. }
DE FALSA LEGATIONE. 6*s*.

Demosthenis Orationes Privatae.
By ARTHUR HOLMES, M.A.
DE CORONA. 5*s*.

Aristophanis Comoediae. *By* W. C. GREEN, M.A.
THE ACHARNIANS AND THE KNIGHTS. 4*s*.
THE WASPS. 3*s*. 6*d*. THE CLOUDS. 3*s*. 6*d*.
An Edition of THE ACHARNIANS AND THE KNIGHTS, revised and especially adapted for use in Schools. 4*s*.

Isocratis Orationes. *By* JOHN EDWIN SANDYS, M.A.
AD DEMONICUM ET PANEGYRICUS. 4*s*. 6*d*.

Persii Satirae. *By* A. PRETOR, M.A. 3*s*. 6*d*.

Homeri Ilias. *By* S. H. REYNOLDS, M.A.
BOOKS I. TO XII. 6*s*.

Terenti Comoediae. *By* T. L. PAPILLON, M.A.
ANDRIA AND EUNUCHUS. With Introduction on Prosody. 4*s*. 6*d*.
Or separately,
ANDRIA. With Introduction on Prosody. 3*s*. 6*d*.
EUNUCHUS. 3*s*.

Herodoti Historia. *By* H. G. WOODS, M.A.
BOOK I., 6*s*. BOOK II., 5*s*.

Horati Opera. *By* J. M. MARSHALL, M.A.
VOL. I.—THE ODES, CARMEN SECULARE, AND EPODES. 7*s*. 6*d*.

Taciti Historiae. *By* W. H. SIMCOX, M.A.
BOOKS I. AND II., 6*s*. BOOKS III., IV., and V., 6*s*.

DIVINITY

MANUALS OF RELIGIOUS INSTRUCTION

Edited by

JOHN PILKINGTON NORRIS, B.D.,

CANON OF BRISTOL, AND EXAMINING CHAPLAIN TO THE BISHOP OF MANCHESTER.

Three Volumes. Small 8vo. 3*s.* 6*d.* each.
Or each Book in Five Parts. 1*s.* each Part.

"Contain the maximum of requisite information within a surprising minimum of space. They are the best and fullest and simplest compilation we have hitherto examined on the subject treated."
Standard.

"Carefully prepared, and admirably suited for their purpose, they supply an acknowledged want in Primary Schools, and will doubtless be in great demand by the teachers for whom they are intended."
Educational Times.

THE OLD TESTAMENT.

By the Rev. E. I. GREGORY, M.A., *Vicar of Halberton.*

PART I. The Creation to the Exodus. PART II. Joshua to the Death of Solomon. PART III. The Kingdoms of Judah and Israel. PART IV. Hebrew Poetry—The Psalms. PART V. The Prophets of the Captivity and of the Return—The Maccabees—Messianic Teaching of the Old Testament.

THE NEW TESTAMENT.

By C. T. WINTER.

PART I. St. Matthew's Gospel. PART II. St. Mark's Gospel. PART III. St. Luke's Gospel. PART IV. St. John's Gospel. PART V. The Acts of the Apostles.

THE PRAYER BOOK.

By JOHN PILKINGTON NORRIS, B.D., *Canon of Bristol, &c.*

PART I. The Catechism to the end of the Lord's Prayer—The Order for Morning and Evening Prayer. PART II. The Catechism, concluding portion — The Office of Holy Baptism — The Order of Confirmation. PART III. The Theology of the Catechism—The Litany—The Office of Holy Communion. PART IV. The Collects, Epistles, and Gospels, to be used throughout the year. PART V. The Thirty-Nine Articles.

The First German Book.

By T. K. ARNOLD, M.A., *and* J. W. FRÄDERSDORFF, Ph.D.
Eighth Edition. 12mo. 5*s*. 6*d*. KEY, 2*s*. 6*d*.

Lessing's Fables.

Arranged in order of difficulty. A First German Reading Book.
By F. STORR, M.A., *Chief Master of Modern Subjects in Merchant Taylors' School, and late Assistant-Master in Marlborough College.*
Crown 8vo. 2*s*. 6*d*.

CONTENTS:—Introduction—Book I., 29 Fables—Book II., 30 Fables—Book III., 30 Fables—Notes—Vocabulary.

"Few words are necessary to explain the plan on which this little book has been composed, the objects which it is intended to serve, and the way in which it is meant to be used.

"I have for several years taken Lessing's Fables as a first German reading-book. The simplicity of the subject-matter and the language, the shortness of the fables, one or more of which can be read through at a sitting, and, not least, the excellence of the style, an excellence which young boys are quite capable of appreciating, seemed to recommend the book as the best possible German primer. The chief drawback I found in using it was the unequal difficulty of the fables, the first book, as it happens, containing by far the hardest. This I have sought to remedy by rearranging them approximately according to their difficulty.

"Most teachers will agree that the chief stumbling-block in the way of learning German is the disproportionate time that a beginner takes in looking out words. I need not here vindicate the principle of Glossaries *versus* Dictionaries. If any are still unconvinced, let them read the preface to Mr. Quick's *Companion to Wilhelm Tell*, or, better still, let them give the glossary system a trial for a single term.

"In so short a volume I have not thought it necessary to adopt Mr. Quick's charmingly simple but (for the composer) repulsively laborious plan of giving each word in the order of the text. Whenever I thought the beginner would be likely not to know what to look out in the glossary I have helped him in a note.

"In the glossary I have been careful to give in each case the primitive meaning of a word as well as the special meanings which occur in the text. As to etymology, I have added cognate English words, except when they were too obvious to require notice, or too remote to come home to an average schoolboy. Quite apart from the value of the knowledge for itself, I have found by experience that etymology is an invaluable aid to a boy's memory in German almost as much as in French. By these additions, no less than by its omissions, I hope that the glossary will prove more serviceable to the schoolboy than the ordinary dictionary.

"The work is meant to be a first German reading-book. Directly a boy has learnt the article, the indicatives of *sein*, *haben*, and *loben*, and perhaps the adjective, I would set him down to the Fables. Let him pick up the grammar as he goes along, and as he requires it, instead of wasting his energy in what must be a distasteful, and what will seem to him an unpractical, study."—*Extract from the Preface.*

FRENCH

The Campaigns of Napoleon.

The Text (in French) from M. THIERS' "*Histoire de la Révolution Française,*" *and* "*Histoire du Consulat et de l'Empire.*" *Edited, with English Notes, for the use of Schools, by* EDWARD E. BOWEN, M.A., *Master of the Modern Side, Harrow School.*

With Maps. Crown 8vo.

ARCOLA. 4*s.* 6*d.* MARENGO. 4*s.* 6*d.*

JENA. 3*s.* 6*d.* WATERLOO. 6*s.*

Selections from Modern French Authors.

Edited, with English Notes and Introductory Notice, by HENRI VAN LAUN, *Translator of Taine's* HISTORY OF ENGLISH LITERATURE.

Crown 8vo. 3*s.* 6*d.* each.

HONORÉ DE BALZAC. H. A. TAINE.

La Fontaine's Fables. Books I. and II.

Edited, with English Notes at the end, for use in Schools, by Rev. P. BOWDEN-SMITH, M.A., *Assistant-Master at Rugby School.*

Small 8vo. 2*s.*

Books IV. to VI. in the Press.

The First French Book.

By T. K. ARNOLD, M.A.

Sixth Edition. 12mo. 5*s.* 6*d.* KEY, 2*s.* 6*d.*

MISCELLANEOUS

At Home and Abroad; or, First Lessons in Geography.

By J. K. LAUGHTON, M.A., F.R.A.S., F.R.G.S., *Mathematical Instructor and Lecturer in Meteorology at the Royal Naval College.*

Crown 8vo. 3*s*. 6*d*.

The First Hebrew Book.

By T. K. ARNOLD, M.A.

Fifth Edition. 12mo. 7*s*. 6*d*. KEY, 3*s*. 6*d*.

The Chorister's Guide.

By W. A. BARRETT, Mus. Bac. Oxon., *Vicar-Choral of St. Paul's Cathedral, Author of "Flowers and Festivals," &c.*

Second Edition. Crown 8vo. 2*s*. 6*d*.

An Introduction to Form and Instrumentation, for the use of Beginners in Composition.

By W. A. BARRETT, Mus. Bac. Oxon., *Vicar-Choral of St. Paul's Cathedral, Author of "Flowers and Festivals," "The Chorister's Guide," &c.*

Crown 8vo. 2*s*. 6*d*.

INDEX

INDEX.

Scenes from Greek Plays. Rugby Edition. By ARTHUR SIDGWICK, M.A., Assistant-Master at Rugby School.

Small 8*vo*. 1*s*. 6*d*. *each*.

ARISTOPHANES.
THE CLOUDS. THE FROGS. THE KNIGHTS. PLUTUS.

EURIPIDES.
IPHIGENIA IN TAURIS. THE CYCLOPS. ION. ELECTRA. ALCESTIS. BACCHÆ. HECUBA.

Homer's Iliad. Edited, with Notes, at the end, for the use of Junior Students, by ARTHUR SIDGWICK, M.A.

Small 8*vo*.

Books I. and II. 2*s*. 6*d*.

Cicero de Amicitiâ. Edited, with Notes and an Introduction, by ARTHUR SIDGWICK, M.A.

Small 8*vo*. 2*s*.

An Introduction to Greek Prose Composition. By ARTHUR SIDGWICK, M.A.

Second Edition. Crown 8*vo*. 5*s*.

A KEY, for the use of Tutors only, 5*s*.

Selections from Lucian. With English Notes. By EVELYN ABBOTT, M.A., LL.D., Fellow and Tutor of Balliol College, Oxford.

Second Edition. Small 8*vo*. 3*s*. 6*d*.

The Elements of Greek Accidence. With Philological Notes. By EVELYN ABBOTT, M.A., LL.D.

Crown 8*vo*. 4*s*. 6*d*.

A Primer of Greek Accidence. For the Use of Schools. By EVELYN ABBOTT, M.A., LL.D.; and E. D. MANSFIELD, M.A., Assistant-Master at Clifton College. With a Preface by JOHN PERCIVAL, M.A., LL.D., President of Trinity College, Oxford.

Crown 8*vo*. 2*s*. 6*d*.

Stories from Ovid in Elegiac Verse. By R. W. TAYLOR, M.A., Head-Master of Kelly College, Tavistock.

New Edition. Crown 8*vo*. 3*s*. 6*d*.

The Anabasis of Xenophon. Edited, with Notes and Vocabularies by R. W. TAYLOR, M.A.

Crown 8*vo*.

Books I. and II. 3*s*. 6*d*.
Books III. and IV. 3*s*. 6*d*.

First Latin Writer. Comprising Accidence, the Easier Rules of Syntax illustrated by copious Examples, and Progressive Exercises in Elementary Latin Prose, with Vocabularies. By G. L. BENNETT, M.A., Head-Master of the High School, Plymouth; formerly Assistant-Master at Ruby School.

Crown 8*vo*. 3*s*. 6*d*.

A KEY, for the use of Tutors only, 5*s*.

Easy Latin Stories for Beginners. With Notes and Vocabularies, by G. L. BENNETT, M.A.

Second Edition. Crown 8*vo*. 2*s*. 6*d*.

A KEY, for the use of Tutors only. 5*s*.

Arnold's Henry's First Latin Book. A New and Revised Edition. 12*mo*. 3*s*. By C. G. GEPP, M.A., Head-Master of King Edward VI. School, Stratford-upon-Avon.

A KEY, for the use of Tutors only, 5*s*.
[The original Edition of this work is still kept on sale.]

Progressive Exercises in Latin Elegiac Verse. By C. G. GEPP, M.A.

Fourth Edition, Revised. Crown 8*vo*. 3*s*. 6*d*. *Tutor's KEY*, 5*s*.

The Æneid of Vergil. Edited, with Notes at the end, by FRANCIS STORR, M.A., Chief Master of Modern Subjects at Merchant Taylors' School, late Scholar of Trinity College, Cambridge.

Crown 8*vo*.

Books I. and II. 2*s*. 6*d*.
Books XI. and XII. 2*s*. 6*d*.